

Affine quotients of supergroups

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Abstract

In this article we consider sheaf quotients of affine superschemes by affine supergroups that act on them freely. The necessary and sufficient conditions for such quotients to be affine are given. If G is an affine supergroup and H is its normal supersubgroup, then we prove that a dur K -sheaf G/\tilde{H} is again affine supergroup. Additionally, if G is algebraic, then a K -sheaf G/\tilde{H} is also algebraic supergroup and it coincides with G/\tilde{H} . In particular, any normal supersubgroup of an affine supergroup is faithfully exact.

Introduction

Let G be an algebraic group defined over an algebraically closed field K of an arbitrary characteristic. A closed subgroup H of G is called *exact* iff the induced functor ind_H^G is exact. A remarkable theorem of Cline-Parshall-Scott says that H is exact iff the quotient G/H is affine iff $K[G]$ is an injective H -module [21]. In the category of affine schemes, well-known Takeuchi's theorem states that the dur K -sheaf (faisceau dur in the terminology from [6]) of right cosets G/\tilde{H} is affine iff $K[G]$ is a faithfully coflat right (or/and left, respectively) $K[H]$ -comodule iff $K[G]$ is an injective cogenerator in the category of right (or/and left, respectively) $K[H]$ -comodules [12, 14]. In this case H is called *faithfully exact*.

In the category of affine supergroups, only the second equivalence of the above Cline-Parshall-Scott theorem has been recently proved in [1]. The definition of a dur K -sheaf (as well as the definition of a K -sheaf or faisceau, see [6]) can be easily adapted to the category of K -functors over commutative superalgebras. Therefore, it is natural to ask whether, for an affine supergroup G and its supersubgroup H , the dur K -sheaf G/\tilde{H} is affine. In the case when G is algebraic, that is $K[G]$ is a finitely generated superalgebra, one can also ask whether the K -sheaf G/\tilde{H} is affine. Notice that in general we only have inclusion $G/\tilde{H} \subseteq G/\tilde{H}$ but if G/\tilde{H} is an affine superscheme, then $G/\tilde{H} = G/\tilde{H}$.

In the present article we consider more general case when G acts on an affine superscheme X freely and we give necessary and sufficient conditions for X/\tilde{G} (and X/\tilde{G} , respectively) to be an affine superscheme. Following [10, 12, 14] we generalize some theorems about Hopf algebras and their coideal subalgebras to Hopf superalgebras and their coideal supersubalgebras. In particular, we obtain a superalgebra version of the above Takeuchi's theorem.

Equipped with these results, we proceed to prove the main theorem of our article. Namely, the fundamental theorem of the algebraic group theory states that if G is an

algebraic group and H is its closed normal subgroup, then G/H is again an algebraic group, see [6, 17, 27]. For any affine supergroup G and its closed normal supersubgroup H we prove that G/\tilde{H} is again an affine supergroup. Moreover, if G is algebraic, then $G/\tilde{H} = \tilde{G}/H$ is also algebraic. It follows that any closed normal supersubgroup of an affine supergroup is faithfully exact.

The article is organized as follows. In the first section necessary definitions and notations concerning super(co)algebras and super(co)modules over them are collected. On top of that, (co)flat and faithfully (co)flat super(co)modules are defined and some well-known results about flat and faithfully flat modules over commutative algebras are generalized to (super)modules over commutative superalgebras. In the second section we introduce K -functors as functors from the category of commutative superalgebras to the category of sets. More attention is paid to the particular case of affine superschemes. We also define (dur) K -sheafs and completions of certain K -functors in the Grothendieck topology of faithfully flat (faithfully flat and finitely presented) coverings. In the third section we define the superspace of distributions of an affine superscheme. This superspace has a natural structure of (cocommutative) Hopf superalgebra provided the original superscheme is a supergroup. Moreover, if $\text{char} K = 0$ and this supergroup is algebraic, then we prove that the corresponding distribution superalgebra is isomorphic to the universal enveloping superalgebra of its Lie superalgebra. In addition, we introduce the notion of *Lie superalgebra functor* by means of superalgebra of dual numbers (see [29]).

In the fourth section we consider an affine superscheme X and a supergroup G that acts on X (on the right) freely. We give necessary and sufficient conditions for a quotient X/\tilde{G} (X/\tilde{G}) to be affine. The main results of the fifth section are Theorem 5.1 and 5.2. Theorem 5.1 says surprisingly more about Hopf supersubalgebras than one would expect. Keeping in mind Takeuchi's theorem it would be quite natural to expect that a Hopf superalgebra is (left and right) faithfully flat supermodule over its Hopf supersubalgebra but it is actually faithfully flat as a module!

In sixth and ninth sections we prove the main result of this article. The principal idea is different from [11, 17] for the following reason. In the category of vector superspaces, an exterior power does not have the same nice properties as in the category of vector spaces. More precisely, if V is a superspace of finite (super)dimension (m, n) and W is its supersubspace of (super)dimension (s, t) , where $t > 0$, then no exterior power $\Lambda^d(W)$ is one-dimensional. In particular, the "naive" or "direct" generalization of Chevalley's theorem is not possible. Instead of exterior powers one can use a *berezinian* $Ber_{s|t}(W)$ of W but it does not solve our problem. In fact, we would have to embed $Ber_{s,t}(W)$ to something like $Ber_{s,t}(V)$ (analogously as $\Lambda^k(U)$ is embedded into $\Lambda^k(L)$, where U is a subspace of a space L and $k = \dim U$) but there is no appropriate definition of such a supermodule. To overcome this obstacle, we construct to any normal supersubgroup some biggest normal supersubgroup over which the quotient is affine. Next step is to prove that the original supersubgroup coincides with this new one. For $\text{char} K = p > 0$ we use some trick with a Frobenius map. The characteristic zero case is much more complicated and we have to introduce a notion of a *pseudoconnected component* of a supergroup to reduce our problem to the case of a finite normal supersubgroup. Besides, we use induction on superdimensions of Lie superalgebras of our supergroups and some properties of adjoint representations. In the final section an example of faithfully exact supersubgroup is given.

This is a Levi supersubgroup of a general linear supergroup. Furthermore, we construct an affine superscheme X on which a finite (odd) unipotent supergroup G acts in a such way, that neither X/\tilde{G} , nor X/G is affine.

In seventh and eighth section a partial answer for the following Brundan's question is given. Let G be an algebraic supergroups and H be its supersubgroup such that H_{ev} is reductive. Is it true that G/\tilde{H} is affine? We show that the answer is positive if $\text{char} K = p > 0$ or G is finite.

1 Super(co)algebras and super(co)modules

We follow definitions and notations from [1, 2] (see also [8]). Let K be a field of characteristic $p \neq 2$. For a K -vector superspace V its *superdimension* is defined as $\text{sdim } V = (\dim V_0, \dim V_1)$. Let A be a (associative) superalgebra over K . Denote by $A - \text{smod}$ (or $\text{smod} - A$, respectively) the category of all left (or right, respectively) A -supermodules with even morphisms. Let X be a left (or right) A -supermodule with \mathbf{Z}_2 -grading $X = X_0 \oplus X_1$. For any $x \in X$ denote by x_0 and x_1 its homogeneous components, that is $x_0 \in X_0, x_1 \in X_1$ and $x = x_0 + x_1$. If $X \in \text{smod} - A, Y \in A - \text{smod}$, then the tensor product $X \otimes_A Y$ has a natural \mathbf{Z}_2 -grading given by $|x \otimes y| = |x| + |y| \pmod{2}$ for $x \in X, y \in Y$.

A superalgebra A has the opposite companion A° whose underlying superspace coincides with A and the multiplication in A° is defined by $a * b = (-1)^{|a||b|}ba$. It is clear that $(A^\circ)^\circ = A$. We have an equivalence of categories $A - \text{smod} \simeq \text{smod} - A^\circ$ given by $M \mapsto M^\circ$ for $M \in A - \text{smod}$, where M° coincides with M as a superspace and a structure of a right supermodule on M° is defined by $m * a = (-1)^{|m||a|}am$ for $a \in A, m \in M$. For any $X \in \text{smod} - A, Y \in A - \text{smod}$, we have an isomorphism $X \otimes_A Y \simeq Y^\circ \otimes_{A^\circ} X^\circ$ given by $x \otimes y \mapsto (-1)^{|x||y|}y \otimes x$ for $x \in X, y \in Y$ which is functorial in X and Y .

A superalgebra A is called *commutative* if any homogeneous elements $a, b \in A$ satisfy $ab = (-1)^{|a||b|}ba$. In particular, $A = A^\circ$ and $A - \text{smod} \simeq \text{smod} - A$. In other words, any one-sided A -supermodule has a canonical structure of a A -superbimodule. Denote by $SAlg_K$ the category of all commutative K -superalgebras with even morphisms. If $\phi : B \rightarrow A, \psi : B \rightarrow C$ are morphisms in $SAlg_K$, then $A \otimes_B C \in SAlg_K$ (see [8]). From now on, all superalgebras are commutative unless otherwise stated.

Let $K[m|n] = K[t_1, \dots, t_m | z_1, \dots, z_n]$ be a free commutative superalgebra with free generators $t_1, \dots, t_m, z_1, \dots, z_n$, where $|t_i| = 0, |z_j| = 1$ for $1 \leq i \leq m, 1 \leq j \leq n$. It can be identified with the symmetric superalgebra $S(V)$ of a superspace V , where $\dim V_0 = m, \dim V_1 = n$ (see [1]).

Lemma 1.1 *Let $R = R_0 \oplus R_1$ be a superalgebra. Then*

- i) If R is finitely generated, then R is noetherian;*
- ii) Every one-sided superideal of R is two-sided;*
- iii) A prime ideal \mathcal{P} of R has a form $\mathcal{P} = \mathcal{P}_0 + R_1$, where \mathcal{P}_0 is a prime ideal of R_0 . If \mathcal{M} is a maximal one-sided ideal of R , then $\mathcal{M} = \mathcal{M}_0 + R_1$, where \mathcal{M}_0 is a maximal ideal of R_0 . In particular, all these ideals are superideals.*

Proof. The statement ii) is obvious. To prove i) we notice that $K[m|n]$ is finitely generated module over $K[t_1, \dots, t_m]$. Finally, iii) holds since RR_1 is a nil ideal.

Remark 1.1 *It is not true that every one-sided ideal in a superalgebra is necessary two-sided. For example, the left ideal $K[1|2](t_1 + z_1)$ is not two-sided.*

Denote by $\sqrt[n]{I}$ the prime radical of a superideal I . It is the intersection of all prime ideals containing I . It is not difficult to see that

$$\sqrt[n]{I} = \{r \in R \mid \exists n, r^n \in I\} = \{r \in R \mid \exists n, r_0^n \in I_0\},$$

where for the last equality we used the obvious formula $r^n = r_0^n + nr_0^{n-1}r_1$.

Let A be a (not necessary commutative) superalgebra. A left supermodule $Y \in A - smod$ (a right supermodule $Y \in smod - A$, respectively) is called *flat* if the functor $X \rightarrow X \otimes_A Y$ ($X \rightarrow Y \otimes_A X$, respectively) is an exact functor from the category $smod - A$ ($A - smod$, respectively) to the category of superspaces.

Next, $Y \in A - smod$ ($Y \in smod - A$, respectively) is called *faithfully flat* if the corresponding functor is faithfully exact, that is the exactness of any sequence $X' \rightarrow X \rightarrow X''$ in $smod - A$ (in $A - smod$, respectively) is equivalent to the exactness of the sequence of superspaces $X' \otimes_A Y \rightarrow X \otimes_A Y \rightarrow X'' \otimes_A Y$ ($Y \otimes_A X' \rightarrow Y \otimes_A X \rightarrow Y \otimes_A X''$, respectively). Using the previous discussion, a left A -supermodule Y is flat (faithfully flat) iff the right A° -supermodule Y° is flat (faithfully flat). If A is commutative, then any supermodule is left flat (left faithfully flat) iff it is right flat (right faithfully flat). Most of standard characterizations of flatness or faithful flatness from [3] can be easily translated to the supercase. We call such translation a *superversion* of the corresponding statement. Proofs of superversions of results from [3] that are not difficult are left to the reader.

Let A be an algebra and let S be a multiplicative set belonging to the center of A . The algebra of fractions $S^{-1}A$ and the left (or right, respectively) $S^{-1}A$ -module of fractions $S^{-1}X \simeq S^{-1}A \otimes_A X$ (or $S^{-1}X \simeq X \otimes_A S^{-1}A$, respectively) for left (or right, respectively) A -module X is defined in the usual way. If A is a superalgebra, X is an A -supermodule and $S \subseteq A_0$, then $S^{-1}X$ is also an A -supermodule with \mathbf{Z}_2 -grading given by $(S^{-1}X)_i = S^{-1}X_i$ for $i = 0, 1$.

Lemma 1.2 *Let A and S be as above. Then*

- i) $S^{-1}A$ is a flat A -module;
- ii) *If central elements $a_1, \dots, a_n \in A$ generate A as an ideal, then the algebra $\prod_{1 \leq i \leq n} A_{a_i}$ is a faithfully flat (left and right) A -module.*

Proof. The first statement is an easy generalization of Theorem 1 from [3], II, §2. To prove the second statement, use Proposition 1 from [3], I, §3 and observe that all powers of the elements a_1, \dots, a_n again generate A as an ideal.

Lemma 1.3 *A superalgebra A is generated by elements a_1, \dots, a_n as a left (or right) ideal iff A is generated by their even components.*

Proof. Assume that $1 = \sum_{1 \leq i \leq n} b_i a_i$. Set $a_{i,k}, b_{i,k} \in A_k$, for $1 \leq i \leq n$ and $k = 0, 1$. Then

$$\sum_{1 \leq i \leq n} b_{i,0} a_{i,0} = 1 - \sum_{1 \leq i \leq n} b_{i,1} a_{i,1}$$

and the element $1 - \sum_{1 \leq i \leq n} b_{i,1} a_{i,1} \in 1 + AA_1$ is invertible.

Corollary 1.1 *If A is a superalgebra and $a_1, \dots, a_n \in A_0$ generate A_0 as ideal, then the superalgebra $\prod_{1 \leq i \leq n} A_{a_i}$ is a faithfully flat (left and right) A -module.*

The spectrum of all maximal ideals of a superalgebra A is denoted by $\text{Max}(A)$. For any $\mathcal{M} \in \text{Max}(A)$ we denote by $N_{\mathcal{M}}$ an *even* localization of an A -supermodule N . More precisely, $N_{\mathcal{M}} = (A_0 \setminus \mathcal{M}_0)^{-1}N$.

In what follows all algebras are superalgebras.

Lemma 1.4 *If $\mathcal{M} \in \text{Max}(A)$, then the algebra $A_{\mathcal{M}}$ is local and $\mathcal{M}A_{\mathcal{M}}$ is its Jacobson radical. In particular, the left (and right) A -module $B = \bigoplus_{\mathcal{M} \in \text{Max}(A)} A_{\mathcal{M}}$ is faithfully flat.*

Proof. If an element $a = a_0 + a_1 \in A$ is such that $a_0 \notin \mathcal{M}_0$, then $a^{-1} = \frac{1}{a_0} - \frac{a_1}{a_0^2} \in A_{\mathcal{M}}$. Next, by Lemma 1.2, the A -module B is flat and $\mathcal{M}B \neq B, B\mathcal{M} \neq B$ for all $\mathcal{M} \in \text{Max}(A)$.

Lemma 1.5 *A morphism of left (or right) A -modules $M \rightarrow N$ is a monomorphism (an epimorphism or an isomorphism, respectively) iff the induced morphism $M_{\mathcal{M}} \rightarrow N_{\mathcal{M}}$ is injective (surjective or bijective, respectively) for any $\mathcal{M} \in \text{Max}(A)$.*

Proof. A word-by-word repetition of the proof of Theorem 1, II, §3, [3] combined with Lemma 1.4.

Lemma 1.6 *A left (or right) A -module M is flat (faithfully flat, respectively) iff $M_{\mathcal{M}}$ is a flat (faithfully flat, respectively) $A_{\mathcal{M}}$ -module for all $\mathcal{M} \in \text{Max}(A)$.*

Proof. The necessary condition is a consequence of Proposition 8, I, §2 and Proposition 4, I, §3 from [3]. For the sufficient condition, assume that $M_{\mathcal{M}}$ is a flat (and, for example, right) A -module for all $\mathcal{M} \in \text{Max}(A)$. If $N_1 \rightarrow N_2$ is an inclusion of left A -modules, combine $(M \otimes_A N)_{\mathcal{M}} \simeq M_{\mathcal{M}} \otimes_A N$ with Lemma 1.5 to obtain that $M \otimes_A N_1 \rightarrow M \otimes_A N_2$ is again an inclusion. Additionally, if $M_{\mathcal{M}}$ is faithfully flat for all $\mathcal{M} \in \text{Max}(A)$, then $MM_{\mathcal{M}} \neq M_{\mathcal{M}}$ implies $\mathcal{M}M \neq M$.

Proposition 1.1 *Let $\phi : A \rightarrow B$ be a morphism of superalgebras and let M be a B – A -bimodule (an A – B -bimodule, respectively) such that $ma = \phi(a)m$ for $m \in M, a \in A_0$ ($am = m\phi(a)$ for $m \in M, a \in A_0$, respectively). Then the following properties are equivalent:*

- i) M is a flat A -module;*
- ii) $M_{\mathcal{N}}$ is a flat A -module for all $\mathcal{N} \in \text{Max}(B)$;*
- iii) $M_{\mathcal{N}}$ is a flat $A_{\mathcal{M}}$ -module for every $\mathcal{N} \in \text{Max}(B)$, where $\mathcal{M} = \phi^{-1}(\mathcal{N})$.*

Proof. Use Lemmas 1.2 and 1.5 to copy the proof of Proposition 15, II, §3, [3].

Corollary 1.2 *If M is a B -supermodule, then the conditions of Proposition 1.1 hold automatically. In particular, the properties i)-iii) are equivalent.*

Let M be a flat left (or right, respectively) A -module over an algebra A and let α be an automorphism of A . Denote by M^{α} an A -module such that $M^{\alpha} = M$ and $a \star m = \alpha(a)m$ (or $m \star a = m\alpha(a)$, respectively) for $a \in A, m \in M$. The corollary after Proposition 13, I, §2 of [3] implies that M is a flat A -module iff M^{α} is a flat A -module.

All necessary definitions and notations concerning supercoalgebras, Hopf superalgebras and supercomodules over them can be found in [1, 2]. If C is a supercoalgebra and V is a left (or right, respectively) C -supercomodule, then a counit of C and a coaction map $V \rightarrow C \otimes V$ (or $V \rightarrow V \otimes C$, respectively) are denoted by ϵ_C and τ_V correspondingly. A comultiplication of C is denoted by δ_C . Additionally, if C is a Hopf superalgebra, then its antipode is denoted by s_C . The category of left (or right, respectively) C -supercomodules with even morphisms is denoted by $C - scomod$ (or $scomod - C$, respectively). If $V \in scomod - C, W \in C - scomod$, then one can define a *cotensor* product

$$V \square_C W = \{x \in V \otimes W | (\tau_V \otimes id_W)(x) = (id_V \otimes \tau_W)(x)\}.$$

A left (or right, respectively) C -supercomodule V is called (*faithfully*) *coflat* if the functor $W \rightarrow W \square_C V$ (or $W \rightarrow V \square_C W$, respectively) is (faithfully) exact, where $W \in scomod - C$ (or $W \in C - scomod$, respectively).

Lemma 1.7 (see A.2.1, [14]) *A right (or left) C -supercomodule V is coflat (faithfully coflat, respectively) iff V is injective (an injective cogenerator, respectively).*

Proof. Let $V \in scomod - C, W \in C - scomod$ and $\dim W < \infty$. The dual superspace W^* has a uniquely defined structure of a right C -supercomodule given by $\sum f_1(w)c'_2 = \sum f(w_1)c_2$, where $\tau_W(w) = \sum c_2 \otimes w_1, \tau_{W^*}(f) = \sum f_1 \otimes c'_2$ for $w \in W, f \in W^*$. We have an isomorphism of superspaces $V \square_C W \rightarrow Hom_C(W^*, V)$ induced by $v \otimes w(f) = f(w)v$ for $v \in V, w \in W, f \in W^*$. Since this isomorphism is functorial in W , we conclude the proof as in [14].

2 K -functors and K -sheafs (faisceaux)

Following the book [4] we call a functor from the category $SAlg_K$ to the category of sets a K -functor. The category of all K -functors is denoted by \mathcal{F} . A K -functor $SSp R$ defined as $SSp R(A) = Hom_{SAlg_K}(R, A)$ for $A \in SAlg_K$ is called an *affine superscheme* (this definition is different from the definition used in [1] since we do not suppose that R is finitely generated). The superalgebra $R \in SAlg_K$ is called a *coordinate superalgebra* of the superscheme $SSp R$. If $X = SSp R$, then R is also denoted by $K[X]$.

Lemma 2.1 (Yoneda's lemma, [4], part I, (1.3)) *For an affine superscheme $SSp R$ and a K -functor X there is a canonical isomorphism $Mor(SSp R, X) \simeq X(R)$ which is functorial in both arguments. In particular, the category $SAlg_K$ is anti-equivalent to the full subcategory of affine superschemes.*

Proof. The statement of the lemma is a partial case of more general theorem about covariant representable functors (see [5], Theorem 1.6). The isomorphism is given by $f \mapsto x_f = f(R)(id_R)$ for $f \in Mor(SSp R, X)$ and the inverse map is given by $x \mapsto f_x$, where $f_x(\alpha) = X(\alpha)(x)$ for $x \in X(R), \alpha \in Hom_{SAlg_K}(R, A)$ and $A \in SAlg_K$.

Corollary 2.1 *The universal property of $A \otimes_B C$ implies a canonical isomorphism*

$$SSp A \times_{SSp B} SSp C \simeq SSp A \otimes_B C.$$

The affine superscheme $\mathbf{A}^{m|n} = SSp K[t_1, \dots, t_m | z_1, \dots, z_n]$ is called $(m|n)$ -affine superspace. It is clear that $\mathbf{A}^{m|n}(B) = B_0^m \oplus B_1^n$ for $B \in SAlg_K$. In particular, $\mathbf{A}^{1|1}(B) = B$ for any superalgebra B .

Let I be a superideal of $R \in SAlg_K$. Denote by $V(I)$ a closed subfunctor of $SSp R$ corresponding to I . By definition, $V(I)(A) = \{\phi \in SSp R(A) | \phi(I) = 0\}$. It is obvious that $V(I) \simeq SSp R/I$. All standard properties of closed subfunctors of affine schemes mentioned in [4], part I, (1.4) are translated to the category of affine superschemes per verbatim.

Let X be an affine superscheme. A functor $Y \subseteq X$ is called *open* if

$$Y(A) = \{x \in X(A) | \sum_{f \in I} Ax(f) = A\}$$

for a subset $I \subseteq K[X]$ and $A \in SAlg_K$. Denote this functor by $D(I)$.

Lemma 2.2 *i) If J is the smallest superideal containing I , then $D(I) = D(J) = D(J_0)$;
ii) Let I and I' be superideals of R . Then $D(I) \subseteq D(I')$ iff $\sqrt[R]{I} \subseteq \sqrt[R]{I'}$. Additionally, $D(I) = D(\sqrt[R]{I}) = D(\sqrt[0]{I_0})$.*

Proof. All statements can be proved by the same trick with a representation of unit as in Lemma 1.3 and by the standard reductions to quotients modulo prime ideals (see [4], part I (1.6)).

An important example of an open subfunctor is a so-called *principal open* subfunctor $X_f = D(\{f\})$ for $f \in K[X]$. It can be checked easily that $X_f = X_{f_0}$ is again an affine superscheme and $K[X_f] = K[X]_{f_0}$. All other properties of open subfunctors mentioned in [4], part I (1.6) are easily translated to the category of affine superschemes.

Let G be a *group* K -functor, that is a K -functor to the category of groups. We say that G acts on a K -functor X on the right (on the left, respectively) if there is a morphism of functors $f : X \times G \rightarrow X$ ($g : G \times X \rightarrow X$, respectively) such that $f(id_X \times \mu) = f(f \times id_G)$ and $f i_E = id_X$ ($g(\mu \times id_X) = g(id_G \times g)$ and $g j_E = id_X$, respectively). Here $\mu : G \times G \rightarrow G$ is a multiplication of G and $i_X : X \rightarrow X \times G$ is defined as $i_X(R)(x) = (x, 1_{G(R)})$ ($j_X(R)(x) = (1_{G(R)}, x)$, respectively) for $x \in X(R)$, $R \in SAlg_K$. From now on we consider any action on right unless otherwise stated.

It is obvious that the category of affine group superschemes (= affine supergroups) is anti-equivalent to the category of commutative Hopf superalgebras (see [1, 4]). If G is an affine supergroup, then denote by ϵ_G, δ_G and s_G the counit, comultiplication and antipode of $K[G]$ correspondingly. If $K[G]$ is finitely generated, then G is called an *algebraic* supergroup. Closed supersubgroups $H \leq G$ are in one-to-one correspondence with Hopf superideals $I_H \subseteq K[G]$ such that $H = V(I_H)$. If X and G are affine, then an action of G on X is uniquely defined by a morphism of superalgebras $\tau : K[X] \rightarrow K[X] \otimes K[G]$ with respect to which $K[X]$ is a right $K[G]$ -supercomodule.

Let V be a vector superspace of superdimension (m, n) . Denote by $GL(V)$ or by $GL(m|n)$ the corresponding general linear supergroup. More precisely, $GL(V)$ is a group K -functor such that for any $B \in SAlg_K$ the group $GL(V)(B)$ consists of all even and B -linear automorphisms of $V \otimes B$. It is not difficult to see that $GL(V)$ is an algebraic supergroup (see [1, 24, 25] for more details). A *linear representation* of a group K -functor G is a morphism of group K -functors $\rho : G \rightarrow GL(V)$. In this case the superspace V

is called *G-supermodule*. If G is an affine supergroup, then V is a G -supermodule iff it is a right $K[G]$ -supercomodule (see [1, 4]). In fact, fix a basis v_1, \dots, v_{m+n} of V such that $|v_i| = 0$ if $1 \leq i \leq m$ and $|v_i| = 1$ otherwise. Set $\tau_V(v_i) = \sum_{1 \leq j \leq m+n} v_j \otimes r_{ji}$ for $1 \leq i \leq m+n$. Then $\rho(g) = (g(r_{ij}))$ for $g \in G(B)$ and $B \in SAlg_K$.

From now on, any group K -functor is affine unless otherwise stated. Let W be a supersubspace of a finitely dimensional G -supermodule V . The stabilizer $Stab_G(W)$ is a group subfunctor defined as $Stab_G(W)(A) = \{g \in G(A) | g(W \otimes 1) \subseteq W \otimes A\}$ for $A \in SAlg_K$. It is easy to see that $Stab_G(W)$ is a closed supersubgroup of G . In fact, without loss of generality one can assume that $v_1, \dots, v_s, v_{m+1}, \dots, v_{m+t}$ for $s \leq m, t \leq n$ is a basis of W . Denote by M the set of indexes $\{1, \dots, s, m+1, \dots, m+t\}$. Then $g \in Stab_G(W)(A)$ iff $g(r_{ji}) = 0$ for all $j \notin M, i \in M$ and $A \in SAlg_K$.

Let R_1, \dots, R_n be a finite family of commutative R -superalgebras with respect to a set of morphisms $\iota_R^{R_i} : R \rightarrow R_i$ in $SAlg_K$. Such a family is called *faithfully flat covering* of R (ff-covering, for short) whenever R -supermodule $R_1 \times \dots \times R_n$ is faithfully flat. We say that R -superalgebra R' is *finitely presented* if $R' \simeq R[m|n]/I$, where $R[m|n] = R \otimes K[m|n]$ and $I \subseteq R[m|n]$ is a finitely generated superideal. It is not difficult to check that R' is a finitely presented R -superalgebra iff $R' \simeq R \otimes_A A[m|n]/I$, where A is a finitely generated supersubalgebra of R (see [6], I, §3). Following [4] we call a ff-covering R_1, \dots, R_n *fppf-covering* if all R_i are finitely presented R -superalgebras.

A K -functor X is called *dur K-sheaf* (or *faisceau dur*) if for any ff-covering R_1, \dots, R_n of a superalgebra R the diagram

$$X(R) \rightarrow \prod_{1 \leq i \leq n} X(R_i) \xrightarrow{\quad} \prod_{1 \leq i, j \leq n} X(R_i \otimes_R R_j)$$

is exact, where the last two maps are induced by morphisms $R_i \rightarrow R_i \otimes_R R_j$ and $R_i \rightarrow R_j \otimes_R R_i$, respectively, defined as $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$ for $a, b \in R_i$. This property is equivalent to the following two conditions. For all $R_1, \dots, R_n, R, R' \in SAlg_K$, where R' is a faithfully flat R -supermodule, there is a canonical bijection $X(\prod_{1 \leq i \leq n} R_i) \simeq \prod_{1 \leq i \leq n} X(R_i)$ and the diagram

$$X(R) \rightarrow X(R') \xrightarrow{\quad} X(R' \otimes_R R')$$

is exact, see [4]. Replacing ff-coverings by fppf-coverings we obtain a definition of a *K-sheaf* (or *faisceau*), cf. [6, 4]. Denote the full subcategory of K -sheafs (dur K -sheafs, respectively) by $\tilde{\mathcal{F}}$ ($\tilde{\tilde{\mathcal{F}}}$, respectively). It is clear that $\tilde{\tilde{\mathcal{F}}} \subseteq \tilde{\mathcal{F}}$ and it can be checked easily that any affine superscheme is a dur K -sheaf, see [4], part I (5.3).

For a K -functor X one can construct an *associated K-sheaf* \tilde{X} and a dur K -sheaf $\tilde{\tilde{X}}$ following the way described in [6], III, §1. We consider a partial case following [4], part I (5.4). Assume that the K -functor X satisfies the following conditions :

- (*) $X(\prod_{1 \leq i \leq n} R_i) \simeq \prod_{1 \leq i \leq n} X(R_i)$ for all $R_1, \dots, R_n \in SAlg_K$;
- (**) $X(R) \rightarrow X(R')$ is an inclusion for arbitrary fppf-covering R' of superalgebra R .

The family of all K -functors satisfying the properties (*) and (**) is closed under direct products. Define a partial order on the set of all fppf-coverings of a superalgebra R by the following rule: $R' \leq R''$ if R'' is a fppf-covering of R' . Clearly, this poset is directed since

$R', R'' \leq R' \otimes_R R''$ for any two fppf-coverings of R . For each superalgebra R define the direct spectrum

$$\mathcal{X}(R) = \{X(R', R) = \ker(X(R') \xrightarrow{\quad} X(R' \otimes_R R')) \mid R' \text{ is a fppf-covering of } R\}$$

with canonical inclusions $X(R', R) \rightarrow X(R'', R)$ for any couple $R' \leq R''$. Observe that $R'' \otimes_R R''$ is a fppf-covering of $R' \otimes_R R'$ and set $\tilde{X}(R) = \varinjlim \mathcal{X}(R)$. The functor \tilde{X} is the required completion of X with respect to the Grothendieck topology of fppf-coverings. A canonical inclusion $\alpha_X : X \rightarrow \tilde{X}$ induces a canonical bijection $Mor(\tilde{X}, Y) \rightarrow Mor(X, Y)$ for any K -sheaf Y . One gets easily that if X is a subfunctor of a K -sheaf Y satisfying $(*)$, then $\tilde{X} \subseteq Y$. Moreover,

$$\tilde{X}(R) = \{y \in Y(R) \mid \text{there is } R' \geq R \text{ such that } Y(\iota_R^{R'})(y) \in X(R')\}.$$

To prove all of the above statements we only need a superversion of Proposition 4 [3], I, §3 together with an additional statement which says that $R = \prod_{1 \leq i \leq n} R_i \leq R'$ iff $R_i = R e_i \leq R' e_i$ for any i , where $e_i = (0, \dots, \underbrace{1}_{i\text{-th place}}, \dots, 0)$ and $1 \leq i \leq n$. Arguments

for dur K -sheafs are the same, except that fppf-coverings are replaced by ff-coverings.

Lemma 2.3 *Let G be a group K -functor that satisfies $(*)$ and $(**)$ for all ff-coverings (fppf-coverings, respectively). Then \tilde{G} (\tilde{G} , respectively) is again a group dur K -sheaf (a group K -sheaf, respectively) and the canonical inclusion $G \rightarrow \tilde{G}$ ($G \rightarrow \tilde{G}$, respectively) is a morphism of group functors.*

Proof. Let $g, g_1, g_2 \in \tilde{G}(A)$. For a suitable ff-covering B of a superalgebra A we set $\bar{g} = \tilde{G}(\iota_A^B)(g)$ and $\bar{g}_i = \tilde{G}(\iota_A^B)(g_i) \in G(B)$ for $i = 1, 2$. The pair (\bar{g}_1, \bar{g}_2) belongs to $(G \times G)(B, A)$ which implies $\bar{g}_1 \bar{g}_2 \in G(B, A)$. Define $g_1 g_2 = \tilde{G}(\iota_A^B)^{-1}(\bar{g}_1 \bar{g}_2) \in \tilde{G}(A)$ and $g^{-1} = \tilde{G}(\iota_A^B)^{-1}(\bar{g}^{-1})$. These definitions do not depend on the choice of B . If $\phi : A \rightarrow C$ is a morphism in $SAlg_K$, B and B' are ff-coverings of A and C , respectively, then $B \otimes_A B'$ is a ff-covering of C and B' , respectively. Denote by α and β the morphisms $B \rightarrow B \otimes_A B'$ and $B' \rightarrow B \otimes_A B'$ given by $\alpha(b) = b \otimes 1$, $\beta(b') = 1 \otimes b'$ for $b \in B, b' \in B'$ and set $x_i = \tilde{G}(\phi)(g_i)$, $\bar{x}_i = \tilde{G}(\iota_C^{B'})(x_i)$ for $i = 1, 2$. Then

$$\tilde{G}(\beta \iota_C^{B'} \phi)(g_1 g_2) = \tilde{G}(\alpha \iota_A^B)(g_1 g_2) = G(\alpha)(\bar{g}_1) G(\alpha)(\bar{g}_2) = G(\beta)(\bar{x}_1) G(\beta)(\bar{x}_2) = \tilde{G}(\beta \iota_C^{B'})(x_1 x_2)$$

and since the map $\tilde{G}(\beta \iota_C^{B'})$ is injective we infer that $\tilde{G}(\phi)(g_1 g_2) = x_1 x_2$. The remaining statements of the lemma are now obvious. The case of K -sheafs is analogous.

3 Superalgebras of distributions and Lie superalgebras

Let X be an affine superscheme. Following [4] we call any element of $Dist_n(X, \mathcal{M}) = (K[X]/\mathcal{M}^{n+1})^*$ a *distribution* on X with support at $\mathcal{M} \in Max(K[X])$ of order $\leq n$ (notice that $Max(K[X])$ is obviously identified with $X(K)$). We have $\bigcup_{n \geq 0} Dist_n(X, \mathcal{M}) = Dist(X, \mathcal{M}) \subseteq K[X]^*$. If $g : X \rightarrow Y$ is a morphism of affine superschemes, then it

induces a morphism of filtered superspaces $dg_{\mathcal{M}} : \text{Dist}(X, \mathcal{M}) \rightarrow \text{Dist}(Y, (g^*)^{-1}(\mathcal{M}))$. In particular, if $X = V(I)$ is a closed supersubscheme of Y , then $\text{Dist}(X, \mathcal{M})$ is identified with a filtered supersubspace $\{\phi \in \text{Dist}(Y, \mathcal{M}) | \phi(I) = 0\}$, where $I \subseteq \mathcal{M}$.

If X is an algebraic supergroup and $\mathcal{M} = \ker \epsilon_X$, then $\text{Dist}(X, \mathcal{M})$ is denoted by $\text{Dist}(X)$. In this case $\text{Dist}(X)$ has a structure of a Hopf superalgebra with a multiplication $\phi\psi(f) = \sum (-1)^{|\psi||f_1|} \phi(f_1)\psi(f_2)$ for $\phi, \psi \in \text{Dist}(X), f \in K[X]$ and $\delta_X(f) = \sum f_1 \otimes f_2$, with a unit ϵ_X and with a counit $\epsilon_{\text{Dist}(X)} : \phi \mapsto \phi(1)$. The comultiplication of $\text{Dist}(X)$ is dual to the multiplication of $K[X]$, cf. [24, 25]. Finally, an antipode $s_{\text{Dist}(X)}$ is defined by $s_{\text{Dist}(X)}(\phi)(f) = \phi(s_X(f))$ for $\phi \in \text{Dist}(X)$ and $f \in K[X]$.

We have $\text{Dist}_k(X)\text{Dist}_l(X) \subseteq \text{Dist}_{k+l}(X)$ for all $k, l \geq 0$, that is the superalgebra $\text{Dist}(X)$ is a filtered algebra. The superspace $\text{Lie}(X) = \{\phi \in \text{Dist}_1(X) | \phi(1) = 0\}$ has a Lie superalgebra structure given by $[\phi, \psi] = \phi\psi - (-1)^{|\phi||\psi|}\psi\phi$. As a Hopf superalgebra, $\text{Dist}(X)$ is cocommutative which means that $\delta_{\text{Dist}(X)}(\phi) = \sum \phi_1 \otimes \phi_2 = \sum (-1)^{|\phi_1||\phi_2|} \phi_2 \otimes \phi_1$. Additionally, each $\text{Dist}_n(X)$ is a supersubcoalgebra of $\text{Dist}(X)$. For arbitrary morphism of algebraic supergroups $g : X \rightarrow Y$ its differential $dg = dg_{\mathcal{M}} : \text{Dist}(X) \rightarrow \text{Dist}(Y)$ is a homomorphism of filtered Hopf superalgebras. In particular, its restriction to $\text{Lie}(X)$ is a homomorphism of Lie superalgebras.

Let L be a Lie superalgebra. Denote by $U(L)$ its (*universal*) *enveloping* superalgebra, see [26]. The superalgebra $U(L)$ is a Hopf superalgebra with a comultiplication defined by $\delta_{U(L)}(x) = x \otimes 1 + 1 \otimes x$ for $x \in L$. Its counit $\epsilon_{U(L)}$ is defined by $\epsilon_{U(L)}(L) = 0$ and its antipode $s_{U(L)}$ is defined by $s_{U(L)}(x) = -x$ for $x \in L$. The antipode $s_{U(L)}$ is an anti-automorphism of $U(L)$ such that $s_{U(L)}(uv) = (-1)^{|u||v|} s_{U(L)}(v)s_{U(L)}(u)$ for $u, v \in U(L)$. The Hopf superalgebra $U(L)$ is obviously filtered and cocommutative.

The inclusion $\text{Lie}(X) \subseteq \text{Dist}(X)$ induces a morphism $g_X : U(\text{Lie}(X)) \rightarrow \text{Dist}(X)$ of superalgebras.

Lemma 3.1 (see Lemma 1.2, II, §6, [6]) *If $\text{char} K = 0$ and X is an algebraic supergroup, then g_X is an isomorphism of Hopf superalgebras.*

Proof. Let ϕ_1, \dots, ϕ_n form a basis of $\text{Lie}(X)$ dual to a homogeneous basis f_1, \dots, f_n of the superspace $\mathcal{M}/\mathcal{M}^2$, where $|\phi_i| = |f_i| = 0$ for $1 \leq i \leq t$ and $|\phi_j| = |f_j|$ for $t+1 \leq j \leq n$.

By induction on l and using formula (3.1) of [24] we infer that

$$\begin{aligned} \psi_1 \dots \psi_l(g_1 \dots g_l) &= \sum_{1 \leq i \leq l} (-1)^{\sum_{1 \leq t < i} |g_i||g_t| + \sum_{1 < t} |g_i||\psi_t|} \psi_1(g_i) \times \\ &\quad \sum_{\substack{\sigma(1)=i \\ \sigma(2) \leq \dots \leq \sigma(l)}} (-1)^{\sum_{2 \leq s < t, \sigma(s) > \sigma(t)} |g_{\sigma(s)}||g_{\sigma(t)}| + \sum_{2 \leq j < k} |\psi_k||g_{\sigma(j)}|} \psi_2(g_{\sigma(2)}) \dots \psi_l(g_{\sigma(l)}) \end{aligned}$$

for $\psi_1, \dots, \psi_l \in \text{Lie}(X)$ and $g_1, \dots, g_l \in \mathcal{M}$. This implies

$$\psi_1 \dots \psi_l(g_1 \dots g_l) = \sum_{\sigma \in S_l} (-1)^{\sum_{s < t, \sigma(s) > \sigma(t)} |g_{\sigma(s)}||g_{\sigma(t)}| + \sum_{j < i} |\psi_i||g_{\sigma(j)}|} \psi_1(g_{\sigma(1)}) \dots \psi_l(g_{\sigma(l)}).$$

In particular,

$$\prod_{1 \leq i \leq n} \phi_i^{s_i} \left(\prod_{1 \leq i \leq n} f_i^{d_i} \right) = \pm \delta_{s_1, d_1} \dots \delta_{s_n, d_n} s_1! \dots s_n!,$$

where $l = s_1 + \dots + s_n = d_1 + \dots + d_n$ and $s_{t+1}, \dots, s_n, d_{t+1}, \dots, d_n \in \{0, 1\}$. Comparison of dimensions shows that g_X is an isomorphism of superalgebras.

By definition, $\delta_{Dist(X)}(\phi)(f_1 \otimes f_2) = \phi(f_1 f_2) = \epsilon_X(f_1) f_2 + f_1 \epsilon_X(f_2)$ for any $\phi \in Lie(X)$, that is $\delta_{Dist(X)}(\phi) = \phi \otimes 1 + 1 \otimes \phi$. The same formula (3.1) of [24] implies that $s_X(f) + f \in \mathcal{M}^2$ for $f \in \mathcal{M}$. In particular, $s_{Dist(X)}(\phi)(f) = \phi(s_X(f)) = -\phi(f)$ for $\phi \in Lie(X)$, that is $s_{Dist(X)}(\phi) = -\phi$. Thus g_X is an isomorphism of filtered Hopf superalgebras.

For $A \in SAlg_K$ let $A[\varepsilon_0, \varepsilon_1]$ be a (commutative) superalgebra of dual numbers. By definition, $A[\varepsilon_0, \varepsilon_1] = \{a + \varepsilon_0 b + \varepsilon_1 c \mid a, b, c \in A\}$, $|\varepsilon_i| = i$, $\varepsilon_i \varepsilon_j = 0$, $i, j \in \{0, 1\}$. We have two morphism of superalgebras $p_A : A[\varepsilon_0, \varepsilon_1] \rightarrow A$ and $i_A : A \rightarrow A[\varepsilon_0, \varepsilon_1]$ defined by $a + \varepsilon_0 b + \varepsilon_1 c \mapsto a$ and $a \mapsto a$ respectively. Define the functor $\mathbf{Lie}(G)$ as

$$\mathbf{Lie}(G)(A) = \ker(G(A[\varepsilon_0, \varepsilon_1]) \xrightarrow{G(p_A)} G(A)), A \in SAlg_K.$$

It is called *Lie superalgebra functor* of G . Let V be a superspace. Define the functor V_a from $SAlg_K$ to the category of vector superspaces, by $V_a(A) = V \otimes A$. The following lemma is obvious (see also [27, 29]).

Lemma 3.2 *There is an isomorphism of abelian group functors $Lie(G)_a \simeq \mathbf{Lie}(G)$ given by*

$$(v \otimes a)(f) = \epsilon_G(f) + (-1)^{|a||f|} \varepsilon_{|v \otimes a|} v(f) a, v \in Lie(G) = (\mathcal{M}/\mathcal{M}^2)^*, a \in A, f \in K[G].$$

If we identify $Lie(G) \otimes A$ with $Hom_K(\mathcal{M}/\mathcal{M}^2, A)$ via $(v \otimes a)(f) = (-1)^{|a||f|} v(f) a$, then the above isomorphism can be represented as

$$u \mapsto \epsilon_G + \varepsilon_0 u_0 + \varepsilon_1 u_1, u \in Hom_K(\mathcal{M}/\mathcal{M}^2, A).$$

Besides, this isomorphism induces the A -supermodule structure on $\mathbf{Lie}(G)(A)$. The supergroup G acts on the functor $\mathbf{Lie}(G)$ by

$$(g, x) \mapsto G(i_A)(g) x G(i_A)(g)^{-1}, g \in G(A), x \in \mathbf{Lie}(G)(A), A \in SAlg_K.$$

This action is called *adjoint* and denoted by \mathbf{Ad} .

Lemma 3.3 *The adjoint action is linear. In particular, it induces a supergroup morphism $G \rightarrow GL(Lie(G))$.*

Proof. Let $u = v \otimes a \in Lie(G) \otimes A$ and $g \in G(A)$. Denote the element $G(i_A)(g)$ by \bar{g} . Then

$$\begin{aligned} \bar{g} u \bar{g}^{-1}(f) &= \sum g(f_1) u(f_2) g(s_G(f_3)) = \epsilon_G(f) + \sum (-1)^{|a||f| + |f_1||v|} \varepsilon_{|u|} g(f_1) v(f_2) g(s_G(f_3)) a = \\ &= (\bar{g}(v \otimes 1) \bar{g}^{-1} a)(f), \end{aligned}$$

where $(\delta_G \otimes 1) \delta_G(f) = (1 \otimes \delta_G) \delta_G(f) = \sum f_1 \otimes f_2 \otimes f_3, f \in K[G]$.

Lemma 3.4 *The differential of \mathbf{Ad} coincides with $-\mathbf{ad}$.*

Proof. Denote $Lie(G)$ by L . We have a commutative diagram

$$\begin{array}{ccc} G(K[\varepsilon_0, \varepsilon_1]) & \xrightarrow{Ad} & GL(L)(K[\varepsilon_0, \varepsilon_1]) \\ \uparrow & & \uparrow \\ Lie(G) & \xrightarrow{d(Ad)} & gl(L). \end{array}$$

Notice that the image of $A \in gl(L)$ in $GL(L)(K[\varepsilon_0, \varepsilon_1])$ is equal to $id_L + \varepsilon_0 A_0 + \varepsilon_1 A_1$. Choose $x, y \in Lie(G)$. Then

$$t = (d(\mathbf{Ad})(x))(y) = (\epsilon_G + \varepsilon_{|x|}x)(\epsilon_G + \varepsilon'_{|y|}y)(\epsilon_G - \varepsilon_{|x|}x),$$

where the product is computed in $GL(L)((K[\varepsilon_0, \varepsilon_1])[\varepsilon'_0, \varepsilon'_1])$. Further,

$$\begin{aligned} t(f) &= \epsilon_G(f) + \varepsilon'_{|y|}(y(f) + \varepsilon_{|x|}(\sum (-1)^{|x||y|} x(f_1)y(f_2)\epsilon_G(f_3) - \epsilon_G(f_1)y(f_2)x(f_3))) = \\ &\quad \epsilon_G(f) + \varepsilon'_{|y|}(y(f) - \varepsilon_{|x|}[y, x](f)). \end{aligned}$$

Following [6], II, §4, we will denote the image of $u \in \mathbf{Lie}(G)(A)$ in $G(A[\varepsilon_0, \varepsilon_1])$ by $e^{\varepsilon_0 u_0 + \varepsilon_1 u_1}$.

4 Quotient K -sheafs

Let G be a group dur K -sheaf and assume that G acts freely on a dur K -sheaf X , that is for any $R \in SAlg_K$ the group $G(R)$ acts freely on $X(R)$. Then the functor $R \rightarrow (X/G)_{(n)}(R) = X(R)/G(R)$ satisfies the properties (*) and (**) for ff-coverings. The proof of this fact can be copied from [4], part I (5.5). Call the above functor $(X/G)_{(n)}$ a *naive* quotient, the dur K -sheaf $(X/\tilde{G})_{(n)}$ a *quotient* dur K -sheaf (of X by G) and denote it by X/\tilde{G} . Then $(X/G)_{(n)} \subseteq X/\tilde{G}$ and there is a canonical G -invariant morphism $\tilde{\pi} : X \rightarrow X/\tilde{G}$. Besides, for any other G -invariant morphism of dur K -sheafs $h : X \rightarrow Z$ there is a unique morphism $v : X/\tilde{G} \rightarrow Z$ such that $h = v\tilde{\pi}$ and these properties define X/\tilde{G} uniquely up to an isomorphism. Analogous statements are valid for K -sheafs and there is an inclusion $X/\tilde{G} \subseteq X/\tilde{G}$.

Let X be an affine superscheme and let G be an affine supergroup acting on X via $f : X \times G \rightarrow X$. If $\tau : K[X] \rightarrow K[X] \otimes K[G]$ is a comorphism dual to f , then $K[X]^G = \{a \in K[X] | \tau(a) = a \otimes 1\}$ is a supersubalgebra of $K[X]$. The embedding $K[X]^G \rightarrow K[X]$ induces a G -invariant morphism $i : X \rightarrow SSp K[X]^G$. In particular, there are uniquely defined morphisms $i' : X/\tilde{G} \rightarrow SSp K[X]^G$ and $i'' : X/\tilde{G} \rightarrow SSp K[X]^G$ such that $i'\tilde{\pi} = i = i''\tilde{\pi}$.

Proposition 4.1 *Suppose that X/\tilde{G} (or X/\tilde{G} , respectively) is an affine superscheme. Then i'' (or i' , respectively) is an isomorphism and $K[X]$ is a faithfully flat $K[X]^G$ -supermodule. If X/\tilde{G} is an affine superscheme and G is algebraic, then $K[X]^G \leq K[X]$.*

Proof. Let $X/\tilde{G} = SSp R$ or $X/\tilde{G} = SSp R$. There is a canonical isomorphism $pr_X \times f : X \times G \simeq X \times_{SSp R} X$ (see [4], part I (5.5)) that is dual to the isomorphism of superalgebras $\phi : K[X] \otimes_R K[X] \simeq K[X] \otimes K[G]$ defined by $a \otimes b \mapsto \sum ab_1 \otimes b_2$, where $\tau(b) = \sum b_1 \otimes b_2$ and $a, b \in K[X]$. Repeating the proof of [4], part I (5.7) we obtain an isomorphism of B -superalgebras $B \otimes_R K[X] \simeq B \otimes K[G]$, where B is either ff-covering of R or $R \leq B$. Using a superversion of Proposition 4 from [3], I, §3 we see that $K[X]$ is a faithfully flat R -supermodule. Thus $K[X]^G \subseteq R \subseteq K[X]$. If G is algebraic, then $B \otimes K[G]$ is a finitely

presented B -superalgebra. A supervision of Lemma 1.4, [6], I implies that $R \leq K[X]$ in the case of K -sheafs. Composition of ϕ and an exact sequence

$$0 \rightarrow R \rightarrow K[X] \xrightarrow{\quad} K[X] \otimes_R K[X]$$

gives $R \subseteq K[X]^G$.

Proposition 4.2 *Assume that R is a supersubalgebra of $K[X]^G$ and the canonical morphism $X \times G \rightarrow X \times_{SSp} R X$ is an isomorphism. If $K[X]$ is a faithfully flat R -supermodule, then $R = K[X]^G$ and $X/\tilde{G} \simeq SSp R$. Additionally, if $R \leq K[X]$, then $X/\tilde{G} \simeq SSp R$.*

Proof. Consider a G -invariant morphism of dur K -sheafs $X \rightarrow Z$. By Yoneda's lemma this morphism is uniquely defined by some element $z \in Z(K[X])$. The G -invariance of this morphism is equivalent to the following property. For any $A \in SAlg_K$ and for arbitrary $\alpha \in X(A), \beta \in G(A)$ we have $Z(\alpha)(z) = Z((\alpha \otimes \beta)\tau)(z)$. Set $A = K[X] \otimes K[G]$ and $\alpha : a \mapsto a \otimes 1, \beta : b \mapsto 1 \otimes b$ for $a \in K[X]$ and $b \in K[G]$. Then $\alpha \otimes \beta = id_{K[X] \otimes K[G]}$. Therefore z belongs to the kernel of

$$Z(K[X]) \xrightarrow[Z(\tau)]{Z(\alpha)} Z(K[X] \otimes K[G]).$$

On the other hand, the above diagram can be identified with

$$Z(K[X]) \xrightarrow[Z(\gamma_2)]{Z(\gamma_1)} Z(K[X] \otimes_R K[X])$$

via bijection $Z(K[X] \otimes_R K[X]) \rightarrow Z(K[X] \otimes K[G])$, where $\gamma_1 : a \mapsto a \otimes 1$ and $\gamma_2 : a \mapsto 1 \otimes a$ for $a \in K[X]$. Setting $Z = A^{11}$ we obtain that $K[X]^G \subseteq R$. Furthermore, if Z is a dur K -sheaf, then z belongs to the image of the map $Z(R) \rightarrow Z(K[X])$ induced by the inclusion $R \rightarrow K[X]$. The above quoted Yoneda's lemma completes the proof. These arguments can be repeated per verbatim for K -sheafs.

5 Coideal supersubalgebras of Hopf superalgebras

Let A be a commutative Hopf superalgebra. Its supersubalgebra $B \subseteq A$ is called a left (or right, respectively) *coideal* iff B is a left (or right, respectively) A -supercomodule. A typical example is as follows. If I is a superideal and a coideal of A , then $C = A/I$ is a superbialgebra and $B = A^C$ (or $B = {}^C A$, respectively) is a left (or right, respectively) coideal, see [12]. Denote by \mathcal{H}^C the category whose objects are right A -supermodules and C -supercomodules simultaneously, together with even morphisms such that $\tau_M(ma) = \sum (-1)^{|c_2||a_1|} m_1 a_1 \otimes c_2 p(a_2)$, where $\delta_A(a) = \sum a_1 \otimes a_2$ and $\tau_M(m) = \sum m_1 \otimes c_2$ for $m \in M, a \in A, M \in \mathcal{H}^C$ and $p : A \rightarrow C = A/I$ is the canonical epimorphism. For example, $A, C \in \mathcal{H}^C$.

Symmetrically, let B be a left coideal supersubalgebra of A . Denote by ${}_B \mathcal{H}$ the category whose objects are left B -supermodules and A -supercomodules simultaneously, together with even morphisms such that $\tau_M(bm) = \sum (-1)^{|b_2||a_1|} b_1 a_1 \otimes b_2 m_2$, where $\delta_A(b) = \sum b_1 \otimes b_2$ and $\tau_M(m) = \sum a_1 \otimes m_2$ for $b \in B, m \in M, M \in {}_B \mathcal{H}$. For example, $B, A \in {}_B \mathcal{H}$.

Lemma 5.1 *Let N be a right A -supermodule, B be a coideal superalgebra of A and $M \in {}_B\mathcal{H}$. The linear map $\xi : N \otimes M \rightarrow N \otimes M$, defined by $\xi(n \otimes m) = \sum n a_1 \otimes m_2$, induces an isomorphism of superspaces $N \otimes_B M \rightarrow N \otimes \overline{M}$, where $\overline{M} = M/B^+M$ and $B^+ = B \cap \ker \epsilon_A$.*

Proof. It is not difficult to see that ξ is an isomorphism of superspaces. The inverse ξ^{-1} of ξ is defined by $\xi^{-1}(n \otimes m) = \sum n s_A(a_1) \otimes m_2$. We have

$$\begin{aligned} \xi(nb \otimes m - n \otimes bm) &= \sum n b a_1 \otimes m_2 - \sum (-1)^{|a_1||b_2|} n b_1 a_1 \otimes b_2 m_2 = \\ &= - \sum (-1)^{|a_1||b_2|} n b_1 a_1 \otimes (b_2 - \epsilon_A(b_2)) m_2 \in N \otimes B^+M \end{aligned}$$

and

$$\xi^{-1}(n \otimes bm) = \sum (-1)^{|a_1||b_2|} n s_A(b_1 a_1) \otimes b_2 m_2 = - \left(\sum n s(a_1 b_1) b_2 \otimes m_2 - \sum n s(a_1 b_1) \otimes b_2 m_2 \right).$$

Lemma 5.2 *Let $p : A \rightarrow C$ be an epimorphism of superbialgebras. For any $N \in \mathcal{H}^C$, $M \in A\text{-scomod}$ the previously defined map ξ induces an isomorphism of superspaces $N^C \otimes M \simeq N \square_C M$.*

Proof. It can be checked easily that $\xi(N^C \otimes M) \subseteq N \square_C M$ and therefore it remains to show that $\xi^{-1}(N \square_C M) \subseteq N^C \otimes M$. Let $\{m_i\}_{i \in I}$ be a homogeneous basis of M and $\tau_M(m_i) = \sum_{k \in I} a_{ik} \otimes m_k$ for $i \in I$. Notice that $|a_{ij}| = |m_i| + |m_j| \pmod{2}$ for $i, j \in I$. The condition $\sum_{i \in I} n_i \otimes m_i \in N \square_C M$ is equivalent to equalities $\tau_N(n_i) = \sum_{k \in I} n_k \otimes p(a_{ki})$ for $i \in I$. Applying ξ^{-1} we see that all we have to check is $u_k = \sum_{i \in I} n_i s_A(a_{ik}) \in N^C$ for all k . But this follows from

$$\begin{aligned} \tau_N(u_k) &= \sum_{i, t, l \in I} (-1)^{|a_{it}||a_{lk}|} n_t s_A(a_{lk}) \otimes p(a_{ti} s_A(a_{il})) = \\ &= \sum_{t, l \in I} (-1)^{|a_{it}||a_{lk}|} n_t s_A(a_{lk}) \otimes p(\epsilon(a_{tl})) = u_k \otimes p(1). \end{aligned}$$

Observe that a superideal AB^+ is also a coideal. It follows from $\delta_A(b) = b \otimes 1 + \sum a_1 \otimes b_2$ for $b \in B, b_2 \in B^+$. Define functors $\Phi : {}_B\mathcal{H} \rightarrow \overline{A}\text{-scomod}$ and $\Psi : \overline{A}\text{-scomod} \rightarrow {}_B\mathcal{H}$ by $\Phi(M) = \overline{M}$, $\Psi(N) = A \square_{\overline{A}} N$. The functor Ψ is right adjoint of Φ by [12]. Adjunctions $u_M : M \rightarrow \Psi\Phi(M) = A \square_{\overline{A}} \overline{M}$, $v_N : \Phi\Psi(N) = \overline{A} \square_{\overline{A}} \overline{N} \rightarrow N$ are defined by $m \mapsto \sum a_1 \otimes \overline{m}_2$ and $\sum \overline{n} \otimes a \mapsto \sum \epsilon_A(a) n$. Symmetrically, one can define functors $\Theta : \text{smod} - B \rightarrow \mathcal{H}^C$ and $\Omega : \mathcal{H}^C \rightarrow \text{smod} - B$ by $\Theta(M) = M \otimes_B A$, $\Omega(N) = N^C$. The functor Θ is left adjoint of Ω by [12]. The corresponding adjunctions $f_M : M \rightarrow \Omega\Theta(M) = (M \otimes_B A)^C$ and $g_N : \Theta\Omega(N) = N^C \otimes_B A \rightarrow N$ are defined by $f_M(m) = m \otimes 1$ and $g_N(n \otimes a) = na$.

Lemma 5.3 *Let $V \in \text{smod} - A$ be a flat B -supermodule. Then $V \otimes_B \Psi(N)$ is isomorphic to $V \otimes N$ via $v \otimes (\sum a \otimes n) \mapsto \sum va \otimes n$, where $v \in V, \sum a \otimes n \in \Psi(N)$.*

Proof. Maps $(id_A \otimes ?) \delta_A \otimes id_N$ and $id_A \otimes \tau_N$ are morphisms of left B -supermodules. They combine to a canonical isomorphism $V \otimes_B \Psi(N) \simeq (V \otimes_B A) \square_{\overline{A}} N$ (see also Proposition 1.3, [13]). On the other hand, $\xi : V \otimes_B A \rightarrow V \otimes \overline{A}$ is an isomorphism of right \overline{A} -supercomodules and its composition with $\xi \otimes id_N$ gives $V \otimes_B \Psi(N) \simeq V \otimes \overline{A} \square_{\overline{A}} N \simeq V \otimes N$.

Lemma 5.4 *Let $V \in A - scomod$ and V be a coflat C -supercomodule. For any B -supermodule T there is an isomorphism $T \otimes V \simeq \Theta(T) \square_C V$ defined by $t \otimes v \mapsto \sum t \otimes c_1 \otimes v_2$, where $\tau_V(v) = \sum c_1 \otimes v_2$.*

Proof. Denote by $p : T \otimes A \rightarrow T \otimes_B A$ the canonical epimorphism of superspaces. A sequence $0 \rightarrow \ker p \rightarrow T \otimes A \xrightarrow{p} T \otimes_B A$ is exact in the category $C - scomod$. In particular, it induces a canonical isomorphism $T \otimes_B (A \square_C V) \rightarrow (T \otimes_B A) \square_C V$. Composition with $id_T \otimes \xi$ gives the isomorphism of the lemma.

Lemma 5.5 *Let A be a superalgebra and let $\phi : M \rightarrow N$ be an epimorphism of free A -supermodules of the same finite rank. Then ϕ is an isomorphism.*

Proof. Denote the rank of both M and N by (r, s) . Choose free generators m_i, n_i for $1 \leq i \leq r + s = n$ of supermodules M and N correspondingly. Additionally, assume that $|m_i| = |n_i| = 0$ if $1 \leq i \leq r$ and $|m_i| = |n_i| = 1$ otherwise. Denote $\phi(m_i) = n'_i$ for $1 \leq i \leq n$ and write $n'_i = \sum_{1 \leq k \leq n} b_{ik} n_k, n_i = \sum_{1 \leq k \leq n} c_{ik} n'_k$ for $1 \leq i \leq n$. Matrices $B = (b_{ik})$ and $C = (c_{ik})$ belong to $Mat_{r,s}(A)_0$ and $CB = E_n$. Considering the last matrix equation modulo the superideal AA_1 , we infer that even diagonal blocks of the matrix C are invertible. By Lemma 1.7.2 of [8], (see also Theorem 3.1, [16]) C is invertible. In particular, the elements n'_i are free generators over A .

Proposition 5.1 *If $M \in \mathcal{H}_B$, then a left B -module M_{B^+} is flat.*

Proof. We follow the same ideas as in the proofs of Lemma 3.1 and of Corollary 3.3 in [10]. First of all, it is enough to consider the case when M is finitely generated. Using the isomorphism $K[G] \otimes_B M \rightarrow K[G] \otimes \overline{M}$ we see that a left $K[G]$ -supermodule $K[G] \otimes_B M$ is free of rank (m, n) , where $m = \dim \overline{M}_0$ and $n = \dim \overline{M}_1$. It is clear that there is an isomorphism of B -supermodules $f : B^m \oplus (B^c)^n \rightarrow M$ which induces an isomorphism modulo B^+ . On the other hand, B_{B^+} is a local algebra. Thus $rad M_{B^+} = (B^+ B_{B^+}) M_{B^+}$ and the localization of f , denoted by $f_{B^+} : B_{B^+}^m \oplus (B_{B^+}^c)^n \rightarrow M_{B^+}$, is an epimorphism (cf. [15], 9.3.5, 9.4.1). Tensoring with $K[G]_{B^+}$ produces an epimorphism $g : K[G]_{B^+}^m \oplus (K[G]_{B^+}^c)^n \rightarrow K[G]_{B^+} \otimes_{B_{B^+}} M_{B^+}$. The $K[G]_{B^+}$ -supermodule $K[G]_{B^+} \otimes_{B_{B^+}} M_{B^+}$ can be naturally identified with $K[G]_{B^+} \otimes_B M \simeq K[G]_{B^+} \otimes \overline{M}$. By Lemma 5.5, the map g is an isomorphism and its composition with $B_{B^+}^m \oplus (B_{B^+}^c)^n \rightarrow (K[G]_{B^+}^c)^n \rightarrow K[G]_{B^+}$ is identified with f_{B^+} .

Lemma 5.6 *A Hopf superalgebra H is a direct union of all of its finitely generated Hopf supersubalgebras.*

Proof. Any finite subset $X \subseteq H$ is contained in a finite dimensional supersubcoalgebra C . Let c_i for $1 \leq i \leq t$ be a homogeneous basis of C . The supersubalgebra generated by c_i and $s_H(c_i)$ for $1 \leq i \leq t$ is obviously a Hopf superalgebra containing X .

Theorem 5.1 *The Hopf superalgebra A is a flat left B -module.*

Proof. Without a loss of generality one can assume that K is algebraically closed and by Lemma 5.6 one can suppose that A is finitely generated. For any $\mathcal{M} \in Max(A)$, the map $T_{\mathcal{M}} : A \rightarrow A$ defined by $T_{\mathcal{M}}(f) = \sum \pi_{\mathcal{M}}(f_1) f_2$, where $\delta_A(f) = \sum f_1 \otimes f_2$ and $\pi_{\mathcal{M}} : A \rightarrow A/\mathcal{M} = K$, is an automorphism of the superalgebra A because $A_1 \subseteq \mathcal{M}$. The inverse of $T_{\mathcal{M}}$ is $T_{s_A(\mathcal{M})}$ because $\pi_{s_A(\mathcal{M})}(f) = \pi_{\mathcal{M}}(s_A(f))$ for $f \in A$.

Next, the map $T_{\mathcal{M}}$ takes B to B and \mathcal{M} to $\mathcal{M}_1 = \ker \epsilon_A$. Using Proposition 5.1, Lemma 1.2 and the left-hand side version of Proposition 8, I, §2 of [3] we see that $A_{B^+} \otimes_A A_{\mathcal{M}_1}$ is a flat left B -module. The canonical epimorphism of left B -modules $A_{B^+} \otimes_A A_{\mathcal{M}_1} \rightarrow A_{\mathcal{M}_1}$ is split. Therefore $A_{\mathcal{M}_1}$ is a flat left B -module. The isomorphism $T_{\mathcal{M}}$ takes this module to $A_{\mathcal{M}}^{\alpha_{\mathcal{M}}}$, where $\alpha_{\mathcal{M}} = T_{\mathcal{M}}|_B$. We complete the proof by combining Proposition 1.1, the remark after Corollary 1.2 and the same reductions as in [10] (see also Lemma 7.1, III, §3 of [6]).

Corollary 5.1 *If B is a Hopf supersubalgebra of A , then A is a faithfully flat (left and right) B -module.*

Proof. By Theorem 5.1 it remains to show that $\mathcal{M}A \neq A$ for all $\mathcal{M} \in \text{Max}(B)$. Since $J = AA_1$ is a Hopf superideal and $J \cap B \subseteq \mathcal{M}$, it suffices to check that $\mathcal{M}(A/J) \neq A/J$. On the other hand, A/J is a Hopf algebra and B/J is its Hopf subalgebra. By Takeuchi's theorem (see [12]), A/J is a faithfully flat B/J -module. The right-hand side statement can be deduced from the right-hand side version of Theorem 5.1.

Let G be an affine supergroup and let H be its closed supersubgroup. The supergroup H is called *faithfully exact* in G if $K[G]$ is a faithfully exact right $K[H]$ -supercomodule. We identify the category of right H -supermodules $\text{mod} - H$ with $K[H] - \text{scomod}$. In particular, the functor $\Psi = K[G] \square_{K[H]} ?$ can be identified with the inducing functor ind_H^G (see [1]).

Proposition 5.2 *The quotient G/\tilde{H} is affine iff $I_H = K[G]R^+$ and $K[G]$ is a faithfully flat R -supermodule, where R is a left coideal supersubalgebra of $K[G]^H$. If G is algebraic, then G/\tilde{H} is affine iff $I_H = K[G]R^+$ and $R \leq K[G]$.*

Proof. Lemma 5.1 and the equality $I_H = K[G]R^+$ guarantees that $G \times H \simeq G \times_{SSp R} G$. Conversely, assume that the quotient G/\tilde{H} (or G/\tilde{H}) is affine. In the following commutative diagram of superalgebras

$$\begin{array}{ccc} K[G] \otimes_R K[G] & \xrightarrow{\xi} & K[G] \otimes \overline{K[G]} \\ & \searrow & \swarrow \\ & K[G] \otimes K[H] & \end{array}$$

the horizontal and the left diagonal arrows are isomorphisms. Thus $\overline{K[G]} \rightarrow K[H]$ is an isomorphism and it remains to refer to Proposition 4.2.

Theorem 5.2 *The following statements are equivalent :*

- i) *Quotient G/\tilde{H} is affine;*
- ii) *H is a faithfully exact supersubgroup of G ;*
- iii) *ind_H^G induces an equivalence of $\text{mod} - H$ with the full subcategory ${}_R\mathcal{H}$;*
- iv) *$K[G]$ is an injective cogenerator in the category $\text{mod} - H$.*

Proof. Combining Lemmas 5.1 – 5.4 and Proposition 5.2 with the proof of Theorems 1 and 2 from [12] we easily obtain that the properties i), ii) and iii) are equivalent to each other. The equivalence of iii) and iv) was proved in Lemma 1.7.

6 Quotients by normal supersubgroups

A group K -subfunctor H of a group K -functor G is said *normal* if $H(A) \trianglelefteq G(A)$ for all $A \in \mathcal{SAlg}_K$. If G is a dur K -sheaf (or K -sheaf, respectively) and H is its normal subsheaf, then Lemma 2.3 implies that \tilde{G}/\tilde{H} (or \tilde{G}/H , respectively) is again group dur K -sheaf (or group K -sheaf, respectively) and the canonical morphism $G \rightarrow \tilde{G}/\tilde{H}$ (or $G \rightarrow \tilde{G}/H$, respectively) is a morphism of group K -functors.

If G is an affine supergroup and H is its closed supersubgroup, then $H \trianglelefteq G$ iff it satisfies one of the following conditions

$$\nu_r(f) = \sum (-1)^{|f_1||f_2|} f_2 \otimes f_1 s_G(f_3) \in I_H \otimes K[G],$$

or

$$\nu_l(f) = \sum (-1)^{|f_1||f_2|} f_2 \otimes s_G(f_1) f_3 \in I_H \otimes K[G],$$

for any $f \in I_H$. The first condition is called *right normality* condition and the second one is called *left normality* condition. These conditions are different in general (say for quantum groups) but for supergroups they are equivalent because s_G is an automorphism of superalgebra $K[G]$ of order two (see [9], 1.5, and [1], §2). The morphism ν_l is a superalgebra morphism, it is dual to the morphism of superschemes $G \times G \rightarrow G$ given by $(g_1, g_2) \mapsto g_2^{-1} g_1 g_2$ for $g_1, g_2 \in G(A)$ and $A \in \mathcal{SAlg}_K$ (which defines a right action of G on itself by conjugations). Symmetrically, ν_r is dual to the morphism $(g_1, g_2) \mapsto g_2 g_1 g_2^{-1}$. From now on, all group K -functors are assumed affine and all group subfunctors are assumed closed unless otherwise stated.

Lemma 6.1 *If $H \trianglelefteq G$, then $\text{Lie}(H)$ is a Lie superideal of $\text{Lie}(G)$.*

Proof. It is clear that $\mathbf{Ad}(G) \subseteq \text{Stab}_{GL(\text{Lie}(G))}(\text{Lie}(H))$. Lemma 3.4 concludes the proof.

Let H be a supersubgroup of G . Denote the normalizer of H in G by $N_G(H)$. By definition,

$$N_G(H)(A) = \{g \in G(A) \mid \text{for any } A\text{-superalgebra } B \ G(\iota_A^B)(g)H(B)G(\iota_A^B)(g)^{-1} = H(B)\}.$$

Proposition 6.1 *The normalizer $N_G(H)$ is a closed supersubgroup.*

Proof. Consider $f \in I_H$. Let $\nu_l(f) = \sum u_1 \otimes u_2$ modulo $I_H \otimes K[G]$, where u_1 are linearly independent modulo I_H . The elements u_2 are called *companions* of f . Let I be a superideal of $K[G]$ generated by companions of all f . Set $B = K[H] \otimes A$ and $\iota_A^B = 1 \otimes id_A$. Then $h = p \otimes 1 \in H(B)$, where p is the canonical epimorphism $K[G] \rightarrow K[H]$. We have

$$(G(\iota_A^B)(g)^{-1} h G(\iota_A^B)(g))(f) = \sum u_1 \otimes g(u_2) = 0.$$

Thus $g(u_2) = 0$. Conversely, if all $g(u_2) = 0$, then $G(\iota_A^B)(g)H(B)G(\iota_A^B)(g)^{-1} \subseteq H(B)$ for any A -superalgebra B . It implies $N_G(H) = V(I + s_G(I))$.

Lemma 6.2 *If $H \trianglelefteq G$, then $R = {}^H K[G] = K[G]^H$ is a Hopf supersubalgebra of $K[G]$.*

Proof. It is enough to observe that $f \in R = K[G]^H$ iff $f(gh) = f(g)$ for all $g \in G(A), h \in H(A)$ and $A \in \text{SAlg}_K$. In particular, $f(hg) = f(gg^{-1}hg) = f(g)$ implies that the antipode s_G induces an automorphism of R .

If $\phi : G \rightarrow H$ and $\psi : L \rightarrow H$ are morphisms of affine supergroups, then $G \times_H L$ is a supersubgroup of $G \times L$. In the case $L \leq H$ we can identify the fiber product $G \times_H L$ with a supersubgroup of G that we call a *preimage* of $\phi^{-1}(L)$. Besides, $K[\phi^{-1}(L)] = K[G]/K[G]\phi^*(I_L)$. In particular, a kernel $N = \ker \phi \leq G$ coincides with $G \times_H E$, where E is the trivial supersubgroup of H . It is also a kernel in the category of supergroups (see Proposition 1.6.1 from [9]). As in [9], we have that $N \trianglelefteq G$ and, moreover, $N \trianglelefteq \phi^{-1}(L)$.

Theorem 6.1 *A supergroup G/\tilde{N} is isomorphic to a supersubgroup of H (the image of ϕ). More precisely, $\text{Im}\phi = \text{SSp } K[H]/\ker \phi^* = \text{SSp } \text{Im}\phi^*$. Thus N is faithfully exact supersubgroup of G . If G and H are algebraic, then $G/\tilde{N} = G/\tilde{N}$.*

Proof. The definition of I_N guarantees that the canonical morphism $G \times N \rightarrow G \times_{\text{Im}\phi} G$ is an isomorphism. Besides, $\text{Im}\phi^* \subseteq K[G]^N$. Proposition 5.2 and Corollary 5.1 imply the first statement. The second statement is deduced easily from Lemma 1.1.

Corollary 6.1 *The canonical epimorphism $\phi^{-1}(L) \rightarrow L \cap \text{Im}\phi$, which is dual to the embedding $K[H]/(I_L + \ker \phi^*) \rightarrow K[G]/K[G]\phi^*(I_L)$, induces an isomorphism $\phi^{-1}(\tilde{L})/N \simeq L \cap \text{Im}\phi$. Moreover, the diagram*

$$\begin{array}{ccc} G/\tilde{N} & \simeq & \text{Im}\phi \\ \uparrow & & \uparrow \\ \phi^{-1}(\tilde{L})/N & \simeq & L \cap \text{Im}\phi \end{array},$$

where the vertical maps are natural embeddings, is commutative.

Proposition 6.2 *The following statements are equivalent :*

- i) *A quotient G/\tilde{H} is affine for any algebraic supergroup G and for any normal supersubgroup H of G ;*
- ii) *Quotient G/\tilde{H} is affine for any affine supergroup G and for any normal supersubgroup H of G .*

Proof. We have to check only the implication i) \rightarrow ii). According to Lemma 5.6, $K[G]$ is a direct union of its finitely generated Hopf supersubalgebras, say $K[G] = \bigcup_{\alpha \in \mathcal{A}} B_\alpha$. Set $I_\alpha = B_\alpha \cap I_H$ for $\alpha \in \mathcal{A}$. By Theorem 5.1 and by i) for any pair $H_\alpha = \text{SSp } B_\alpha/I_\alpha \trianglelefteq G_\alpha = \text{SSp } B_\alpha$ we obtain that $B_\alpha = K[G_\alpha]$ is a faithfully flat (left and right) $R_\alpha = K[G_\alpha]^{H_\alpha}$ -module and $I_\alpha = B_\alpha R_\alpha^+$. It is clear that $\bigcup_{\alpha \in \mathcal{A}} R_\alpha = R = K[G]^H$. By Lemma 7.1, III, §3 of [6], $K[G]$ is faithfully flat (left and right) R -module. It remains to observe that $I_H = \bigcup_{\alpha \in \mathcal{A}} I_\alpha = \bigcup_{\alpha \in \mathcal{A}} B_\alpha R_\alpha^+ = K[G]R^+$.

Remark 6.1 *If G is algebraic, $H \trianglelefteq G$ and G/\tilde{H} is affine, then $G/\tilde{H} = G/\tilde{H}$. In fact, $I_H = K[G]R^+$, where $R = K[G]^H$. By Lemma 1.1 the superideal I_H is finitely generated. Moreover, it is generated by some finite subset from R^+ . By Lemma 5.6 this subset is contained in a finitely generated Hopf supersubalgebra $B \subseteq R$. Using Propositions 5.2 and 4.2 we see that $B = R$.*

Proposition 6.3 *Let G be an algebraic supergroup and $H \leq G$. There is a linear representation $\phi : G \rightarrow GL(V)$ such that $\phi : G \simeq \text{Im}\phi$ and $\phi|_H : H \simeq \text{Stab}_G(W)$ for a suitable supersubspace $W \subseteq V$.*

Proof. There is a finite dimensional supersubcomodule $V \subseteq K[G]$ containing all generators of $K[G]$ as well as all generators of I_H . Let v_1, \dots, v_{m+n} be a basis of V such that $|v_i| = 0$ for $1 \leq i \leq m$ and $|v_i| = 1$ otherwise. Additionally, assume that $v_1, \dots, v_s, v_{m+1}, \dots, v_{m+t}$ for $s \leq m$ and $t \leq n$ is a basis of $W = I_H \cap V$. We have a morphism of supergroup $\phi : G \rightarrow GL(V)$ defined by $g \mapsto (g(r_{ij}))$ for $g \in G(A)$ and $A \in \text{SAlg}_K$, where $\tau_V(v_i) = \delta_G(v_i) = \sum_{1 \leq j \leq m+n} v_j \otimes r_{ji}$ for $1 \leq i \leq m+n$. If $M = \{1, \dots, s, m+1, \dots, m+t\}$ and $i \in M, j \notin M$, then $r_{ji} \in I_H$. Superalgebra $\text{Im}\phi^*$ is generated by the elements r_{ij} and by the multiplicative set generated by determinants of even blocks of the matrix (r_{ij}) . On the other hand, $v_i = \sum_{1 \leq j \leq m+n} \epsilon_G(v_j) r_{ji}$ for every i . Thus $\text{Im}\phi^* = K[G]$ and $\ker \phi = E$. Finally, if $i \in M$, then $v_i = \sum_{j \notin M} \epsilon_G(v_j) r_{ji}$. In other words, $\phi(H) = \text{Stab}_G(W)$.

Proposition 6.4 *Let G be a group K -sheaf and $N_1 \leq N_2$ be group subsheafs of G . If $N_1 \trianglelefteq G$, then $H = N_2/N_1$ is a group subsheaf of $M = G/\tilde{N}_1$ and $\tilde{M}/H \simeq G/\tilde{N}_2$. Additionally, if $N_2 \trianglelefteq G$, then $H \trianglelefteq M$ and the last isomorphism is an isomorphism of group sheafs. Analogous statements are valid for K -sheafs.*

Proof. It is an easy consequence of the universal property of quotients combined with Lemma 2.3.

Now we can formulate and prove the main result of this paper.

Theorem 6.2 *If G is an affine supergroup and N is a normal supersubgroup of G , then G/\tilde{N} is again an affine supergroup.*

Let G be an affine supergroup and N is a normal supersubgroup of G . By Proposition 6.2 one can assume that G is algebraic. Define the supersubgroup $\overline{N} \leq G$ in such way that $I_{\overline{N}} = K[G]R^+$, where $R = K[G]^N$. By Lemma 6.2 we have $N \leq \overline{N} \trianglelefteq G$ and $G/\tilde{\overline{N}} \simeq \text{SSp } R$. It remains to prove that $N = \overline{N}$.

Lemma 6.3 *The superalgebra $K[\overline{N}]^N$ coincides with K .*

Proof. The canonical isomorphism $K[G] \otimes_R K[G] \rightarrow K[G] \otimes K[\overline{N}]$ is an isomorphism of right $K[N]$ -supercomodules. Consider the following exact sequence of R -supermodules

$$0 \rightarrow R \rightarrow K[G] \xrightarrow{\phi} K[G] \otimes K[N],$$

where $\phi(a) = \sum a_1 \otimes \overline{a_2} - a \otimes \overline{1}$ for $a \in K[G]$ and $\delta_G(a) = \sum a_1 \otimes a_2$. Using Theorem 5.1 we obtain $K[G] \otimes_R R = (K[G] \otimes_R K[G])^N \simeq K[G] \otimes K[\overline{N}]^N$ which implies $K[\overline{N}]^N = K$.

From now on we assume that G is algebraic and $K[G]^N = K$ unless stated otherwise. Without a loss of generality one can assume that K is algebraically closed. Up to the end of this section $\text{char } K = p > 0$.

The radical r of the superalgebra $K[G]$ is a Hopf superideal. In fact, a superalgebra $K[G]/r \otimes K[G]/r$ is reduced as the coordinate algebra of an affine variety $\text{Max}(K[G]) \times \text{Max}(K[G])$. A supergroup G_{red} corresponding to the Hopf superideal r is pure even. In other words, it is an affine group (= affine group scheme). Besides, $G_{\text{red}} \leq G_{\text{ev}}$, where $I_{G_{\text{ev}}} = K[G]K[G]_1$. It is clear that $U = N \cap G_{\text{red}} \trianglelefteq G_{\text{red}}$.

Lemma 6.4 *We have $K[G_{red}]^{N_{red}} = K$. In particular, $G_{red} \leq U$ or equivalently, $U = N_{red}$.*

Proof. Assume that an element $f \in K[G]_0$ represents a N_{red} -invariant in $K[G_{red}]$, that is $\delta_G(f) - f \otimes 1 \in r \otimes K[G] + K[G] \otimes (r + I_N)$. Since r is a nilpotent ideal, it follows that for a sufficiently large integer $M > 0$ we have $\delta(f^{p^M}) - f^{p^M} \otimes 1 \in K[G] \otimes I_N$. In particular, $f^{p^M} = a \in K$. Thus $f = b + x$, where $b \in K, b^{p^M} = a$ and $x \in r$. The second statement follows from Theorem 4.3 of [11].

According to Proposition 6.3 we can write $G \leq GL(V)$ and $N = Stab_G(W)$ for suitable supersubspaces $W \subseteq V$. Using the notations from Proposition 6.3 one can depict a matrix from $GL(V)(F)$ for $F \in SAlg_K$ as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A = (A_{ij}) \in GL_m(F_0)$, $B = (B_{ij}) \in M_{m \times n}(F_1)$, $C = (C_{ij}) \in M_{n \times m}(F_1)$ and $D = (D_{ij}) \in GL_n(F_0)$ for $i, j = 1, 2$, where the blocks A_{ij} and B_{ij} (C_{ij} and D_{ij} , respectively) have s rows if $i = 1$, and $m - s$ rows if $i = 2$ (t rows if $i = 1$, and $n - t$ rows if $i = 2$, respectively). Symmetrically, the blocks A_{ij} and C_{ij} (B_{ij} and D_{ij} , respectively) have s columns if $j = 1$, and $m - s$ columns if $j = 2$ (t columns if $j = 1$, and $n - t$ columns if $j = 2$, respectively). The supergroup $S(F) = (Stab_{GL(V)}(W))(F)$ consists of all matrices with $A_{21} = 0, B_{21} = 0, C_{21} = 0$ and $D_{21} = 0$.

The open subfunctor $GL(V)_f$, where $f = \det(A_{11})\det(D_{11})$, contains S . Let U be a closed supersubscheme of $GL(V)_f$ defined by equations $A_{11} = E_s, A_{22} = E_{m-s}, D_{11} = E_t, D_{22} = E_{n-t}, B_{11} = 0, B_{12} = 0, B_{22} = 0, C_{11} = 0, C_{12} = 0, C_{22} = 0, D_{12} = 0$ and $A_{12} = 0$. It is obvious that U is an (unipotent) supersubgroup of $GL(V)$.

Proposition 6.5 *We have an isomorphism of superschemes $\psi : GL(V)_f \simeq U \times S$ commuting with the right action of S given by multiplication.*

Proof. The above isomorphism $\psi : GL(V)_f \rightarrow U \times S$ is defined by the rule

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \times \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix},$$

where

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in U, \begin{pmatrix} A'' & B'' \\ C'' & D'' \end{pmatrix} \in S,$$

and

$$A_{11} = A''_{11}, A_{12} = A''_{12}, B_{11} = B''_{11}, B_{12} = B''_{12},$$

$$C_{11} = C''_{11}, C_{12} = C''_{12}, D_{11} = D''_{11}, D_{12} = D''_{12},$$

$$\begin{pmatrix} A'_{21} & B'_{21} \end{pmatrix} = \begin{pmatrix} A_{21} & B_{21} \end{pmatrix} \begin{pmatrix} A_{11} & B_{11} \\ C_{11} & D_{11} \end{pmatrix}^{-1},$$

$$\begin{pmatrix} C'_{21} & D'_{21} \end{pmatrix} = \begin{pmatrix} C_{21} & D_{21} \end{pmatrix} \begin{pmatrix} A_{11} & B_{11} \\ C_{11} & D_{11} \end{pmatrix}^{-1},$$

$$A''_{22} = A_{22} - A'_{21}A_{12} - B'_{21}C_{12}, B''_{22} = B_{22} - A'_{21}B_{12} - B'_{21}D_{12},$$

$$C''_{22} = C_{22} - C'_{21}A_{12} - D'_{21}C_{12}, D''_{22} = D_{22} - C'_{21}B_{12} - D'_{21}D_{12}.$$

The inverse morphism is just the multiplication map.

Lemma 6.5 *The supergroup G is a closed supersubscheme of $GL(V)_f$.*

Proof. Since $N \subseteq GL(V)_f$, we see that the image of f in $K[G]$ is invertible modulo the superideal I_N . On the other hand, $I_N \subseteq r$ and therefore, $f \in K[G]^*$.

Proposition 6.6 *The supergroup G coincides with N .*

Proof. The naive quotient morphism $GL(V)_f \rightarrow GL(V)_f/S$ can be identified with the composition of ψ and the projection $U \times S \rightarrow U$. In particular, $GL(\tilde{V})_f/S = GL(V)_f/S \simeq U$. The induced morphism $\pi : G \rightarrow GL(V)_f/S$ is dual to the composition of the embedding $K[U] \otimes 1 \rightarrow K[U] \otimes K[S]$ and the epimorphism $K[U] \otimes K[S] \rightarrow K[G]$. The last epimorphism is a morphism of right $K[N]$ -supercomodules. Since $K[U] \otimes 1 = (K[U] \otimes K[S])^S$ it follows that $Im\pi^* = K$, that is $G \subseteq S$.

7 Quotients of finite supergroups

Let G be an affine supergroup and let H, N be supersubgroups of G such that H normalizes N . Denote the semi-direct product of H and N by $H \dot{\times} N$. More precisely, $(H \dot{\times} N)(A) = H(A) \times N(A)$ and $(h, n)(h', n') = (hh', h'^{-1}nh'n')$ for any $h, h' \in H(A), n, n' \in N(A), A \in SAlg_K$. We have a natural morphism $g : H \dot{\times} N \rightarrow G, (h, n) \mapsto hn$. By Theorem 6.1 the image of g is a closed supersubgroup of G which is denoted by HN . The construction in Section 2 yields for any $A \in SAlg_K$

$$HN(A) = \{g \in G(A) | \text{there exists ff-covering } B \text{ of } A \text{ such that } G(\iota_A^B)(g) \in H(B)N(B)\}.$$

For example, assume that N is a kernel of an epimorphism $\pi : G \rightarrow L$. The preimage $\pi|_H^{-1}(\pi|_H(H))$ coincides with HN . Theorem 6.1 and Corollary 6.1 imply that $I_{HN} = K[G](\pi^*(\ker \epsilon_L) \cap I_H)$.

Proposition 7.1 (see [4], part I, (6.2)) *The quotient HN/\tilde{H} is isomorphic to $N/(\tilde{N} \cap H)$.*

Proof. The image of the canonical inclusion $(N/(N \cap H))_{(n)} \rightarrow (HN/H)_{(n)}$ is dense with respect to the Grothendieck topology of ff-coverings. In fact, if $gH(A) \in HN(A)/H(A)$, then there is a ff-covering B of A such that $g'H(B) = h''n''H(B) = h''n''(h'')^{-1}H(B)$, where $g' = G(\iota_A^B)(g), h'' \in H(B), n'' \in N(B)$. Therefore,

$$N/(\tilde{N} \cap H) = (N/(\tilde{N} \cap H))_{(n)} = (HN/\tilde{H})_{(n)} = HN/\tilde{H}.$$

Lemma 7.1 *Let A be a finitely generated (commutative) superalgebra and I be a nilpotent superideal of A . If $\dim A/I$ is finite, then A is finite dimensional.*

Proof. Since A_1 is a finitely generated A_0 -module, all we have to check is that $\dim A_0 < \infty$. Denote by V a finite dimensional subspace of A_0 such that $V + I_0 = A_0$. Choose a non-negative integer k such that $I^{k+1} = 0$. Let $I_0 = \sum_{1 \leq i \leq l} A_0 z_i$. For any $a \in A_0$ we have

$$a = v + \sum_{1 \leq i \leq l} a_i z_i, \text{ where } v \in V \text{ and } a_i \in A_0.$$

Repeating this procedure for the coefficients a_i we obtain

$$a = \sum_{0 \leq t \leq lk} \sum_{1 \leq i_1, \dots, i_t \leq l} v_{i_1, \dots, i_t} z_{i_1} \cdots z_{i_t},$$

where $v, v_{i_1, \dots, i_t} \in V$. Therefore $\dim A_0 \leq \frac{l^{lk+1}-1}{l-1} \dim V$.

Corollary 7.1 *An algebraic supergroup G is finite iff G_{red} is finite or iff G_{ev} is finite.*

For any finite supergroup G denote by $|G|$ the dimension of $K[G]$. We call $|G|$ an *order* of G . By Lemma 6.2 of [27] there are pairwise-orthogonal idempotents $e_1, \dots, e_n \in K[G]_0$ such that $\sum_{1 \leq i \leq n} e_i = 1$ and each $K[G]e_i$ has a unique (nilpotent) maximal ideal $re_i, r = \text{rad}K[G]$. Without loss of generality one can assume that $\epsilon_G(e_1) = 1$ and $\epsilon_G(e_i) = 0$ for $i \geq 2$. A supersubalgebra of $K[G]$, generated by e_1, \dots, e_n , is denoted by B .

Lemma 7.2 *Any idempotent of $K[G]$ belongs to B .*

Proof. Let $x = x_0 + x_1$ be an idempotent in $K[G]$. The equality $x^2 = x$ implies that x_0 is also an idempotent and $2x_0x_1 = x_1$. Multiplying the last equality by x_0 we obtain that $4x_0x_1 = x_1$, hence $x_1 = 0$. Any idempotent xe_i belongs to $K[G]e_i$. In particular, $xe_i = \alpha e_i + ye_i$, where $\alpha = 0, 1$ and $y \in r$. On the other hand, $(x - \alpha)e_i$ is again an idempotent that equals to a nilpotent element ye_i . Hence any xe_i is either zero or it is equal to e_i . Therefore $x = \sum_{1 \leq i \leq n} xe_i \in B$.

Lemma 7.3 *The algebra B is a Hopf (super)subalgebra.*

Proof. We have to check that B is a (super)subcoalgebra. The radical of the superalgebra $A = K[G] \otimes K[G]$ equals to $J = r \otimes K[N] + K[N] \otimes r$. Elements $e_i \otimes e_j$ are pairwise orthogonal idempotents and their sum equals $1 \otimes 1$. Furthermore, $Ae_i \otimes e_j / Je_i \otimes e_j \simeq K \otimes K \simeq K$. Applying Lemma 7.6 to the finite supergroup $G \times G$ we conclude that $\delta_G(B) \subseteq B \otimes B$.

Consider the natural epimorphism of supergroups $G \rightarrow SSp B$ and denote by $G^{(0)}$ its kernel. We can consider $G^{(0)}$ as a *connected component* of G . The equivalent definition will be given in Section 9.

Assume that there is a finite supergroup G and its supersubgroup H such that G/\tilde{H} is not affine. The pair (G, H) is said *bad*. A bad pair defines a vector $v_{G,H} = (|G|, |G| - |H|) \in \mathbf{N}^2$. Order vectors from \mathbf{N}^2 lexicographically from left to right. Choose a bad pair (G, H) whose $v_{G,H}$ is minimal. Denote $K[G]^H$ by R .

By Theorem 5.2(ii) the property to be a faithfully exact supersubgroup is transitive. Therefore, there is not any supersubgroup H' such that $H < H' < G$. On the other hand, the superideal $K[G]R^+ + K[G]s_G(R^+)$ is contained in I_H and it is Hopf one. There is a supersubgroup H' such that $I_{H'} = K[G]R^+ + K[G]s_G(R^+)$. Since $H \leq H'$ it follows that either $H = H'$, that is $I_H = K[G]R^+ + K[G]s_G(R^+)$, or $G = H'$, that is $R = K$.

Assume that $E < G^{(0)} < G$. The minimality of $v_{G,H}$ and Proposition 7.1 imply that $HG^{(0)} < G$ and therefore, $H = HG^{(0)}$. Thus $G^{(0)} \leq H$. In the same way, by Proposition 6.4 we see that G/\tilde{H} is affine. Finally, if $G^{(0)} = E$, then G is pure even and G/\tilde{H} is always affine. The remaining case $G = G^{(0)}$ means that $K[G]$ is a local superalgebra with $\ker \epsilon_G = r$.

Lemma 7.4 *The superalgebra $K[G]$ is a free R -module.*

Proof. By Theorem 5.1 $K[G]$ is a flat R -module. It remains to notice that R^+ is a maximal nilpotent ideal of R and use Corollary 2.1 [3], II, §3.

Lemma 7.5 *Let L be an affine supergroup and N be an its supersubgroup such that $I_N = K[L]T^+ + K[L]s_G(T^+)$, $T = K[L]^N$. Then, the canonical morphism $L \rightarrow SSp\ T$ induces an inclusion $L/\tilde{N} \subseteq SSp\ T$.*

Proof. One has to check that the induced morphism $(L/N)_{(n)} \rightarrow SSp\ T$ is injective. Let $g_1, g_2 \in L(A)$, $A \in SAlg_K$, and assume that $g_1|_T = g_2|_T$. For a given $t \in T^+$ we have $\delta_L(t) = t \otimes 1 + \sum h_1 \otimes t_2$, where $t_2 \in T^+$. By definition,

$$(g_1^{-1}g_2)(t) = g_1(s_L(t)) + \sum g_1(s_L(h_1))g_2(t_2) = g_1(s_L(t)) + \sum g_1(s_L(h_1))g_1(t_2) =$$

$$g_1(s_L(t) + \sum s_L(h_1)t_2) = g_1(\epsilon_L(t)) = 0.$$

Analogously,

$$(g_1^{-1}g_2)(s_L(t)) = g_2(s_L(t)) + \sum (-1)^{|h_1||t_2|} g_1(t_2)g_2(s_L(h_1)) =$$

$$g_2(s_L(t)) + \sum (-1)^{|h_1||t_2|} g_2(t_2)g_2(s_L(h_1)) = g_2^{-1}(\epsilon_L(t)) = 0.$$

It follows that $g_1^{-1}g_2(I_N) = 0$, that is $g_1^{-1}g_2 \in N(A)$.

If $H = H'$, then Lemma 7.5 implies that the morphism of K -functors $G \times H \rightarrow G \times_{SSp\ R} G$ defined as $(g, h) \mapsto (g, gh)$, $g \in G(A)$, $h \in H(A)$, $A \in SAlg_K$, is an isomorphism. Combining Proposition 4.2 with Lemma 7.4 we obtain that $G/\tilde{H} \simeq SSp\ R$!

Let $R = K$. Since the ideal I_H is nilpotent, one can repeat the arguments from Propositions 6.5 and 6.6 to conclude that $G = H$. Resuming all the above we obtain

Theorem 7.1 *Let G be a finite supergroup and $H \leq G$. Then, $G/\tilde{H} = SSp\ K[G]^H$.*

Remark 7.1 *In the conditions of the above theorem we have $G/\tilde{H} = G/\tilde{H}$. In fact, $K[G]^H$ is obviously noetherian and it remains to use Proposition 4.2.*

8 Brunden's question, $\text{char } K = p > 0$

Let G be an algebraic supergroup and H be its supersubgroup such that H_{ev} is reductive. It is equivalent to say that H_{ev} is geometrically reductive [18, 19] and the last property is kept by taking quotients and normal subgroups [20]. As above, we assume that K is algebraically closed. Let A be a superalgebra. Denote by $A^{(n)}$ the superalgebra that coincides as a ring with A , but where each $a \in K$ acts as $a^{p^{-n}}$ does on A (see [4]).

Lemma 8.1 *(see [2], 3.1(a)) If A is a Hopf superalgebra and $\text{char } K = p > 0$, then the linear map $F^n : x \mapsto x^{p^n}$ (Frobenius morphism) is a Hopf superalgebra morphism $A^{(n)} \rightarrow A$.*

Proof. The identity $a^{p^n} = a_0^{p^n}$ for $a \in A$ implies that F^n is a superalgebra morphism. Since δ_A and s_A are superalgebra morphisms, the equations $s_A F^n = F^n s_A$ and $\delta_A F^n = (F^n \otimes F^n) \delta_A$ follow easily.

Denote by $f_n : G \rightarrow SSp K[G]^{(n)}$ the morphism of supergroups dual to $F^n : K[G]^{(n)} \rightarrow K[G]$. The kernel $G_n = \ker f_n$ is called the n -th *infinitesimal* supersubgroup. By Theorem 6.1, G_n is faithfully exact supersubgroup of G . Besides, $G/\tilde{G}_n \simeq SSp K[G]_0^{p^n}$. If G is algebraic, then any G_n is finite.

Lemma 8.2 *Let L be an algebraic supergroup. For sufficiently large $t \geq 1$ the epimorphism $L \rightarrow L/L_t$ induces an epimorphism $L_{ev} \rightarrow L/L_t$. In particular, if L_{ev} is reductive, then L/L_t is also reductive.*

Proof. One has to check that $K[L]_0^{p^t} \cap K[L]_1^2 = 0$ for some $t \geq 1$. Let I be a radical of $K[L]_0$. The algebra $K[L]_0$ is noetherian. It follows that $I^s = 0$ for some $s \geq 1$. If W is a complement of vector subspace I to $K[L]_0$, then $K[L]_0^{p^t} = W^{p^t}$, provided $p^t \geq s$. It remains to notice that $K[L]_1^2 \subseteq I$ and $W^{p^t} \cap I = 0$.

Proposition 8.1 *The quotient G/\tilde{H} is affine.*

Proof. Consider the supersubgroup HG_t . By Lemma 3.1 one can assume that $H_{ev} \rightarrow H/\tilde{H}_t = HG_t/\tilde{G}_t$ is an epimorphism. Combine with Corollary 4.5 from [21] we see that the quotient of G/\tilde{G}_t over the supersubgroup HG_t/\tilde{G}_t is affine. Since the property to be a faithfully exact supersubgroup is transitive we refer to Theorem 7.1 and Proposition 7.1 to conclude the proof.

9 Quotients by normal supersubgroups, $\text{char} K = 0$

An algebraic supergroup G is called *pseudoconnected* if $\bigcap_{n \geq 0} \mathcal{M}^n = 0$, where $\mathcal{M} = \ker \epsilon_G$.

Lemma 9.1 *Let G be an algebraic supergroup. The superideal $I = \bigcap_{n \geq 0} \mathcal{M}^n$ is a Hopf superideal and the supersubgroup $G^{[0]} = V(I)$ is normal and connected.*

Proof. By definition, $s_G(\mathcal{M}) = \mathcal{M}$. It remains to check that I is a coideal and $\nu_l(I) \subseteq I \otimes K[G]$. The trivial supersubgroup is obviously normal. In particular, $\nu_l(\mathcal{M}) \subseteq \mathcal{M} \otimes K[G]$ which implies $\nu_l(\mathcal{M}^n) \subseteq \mathcal{M}^n \otimes K[G]$ for all $n \geq 0$ and we conclude that $\nu_l(I) \subseteq I \otimes K[G]$. Furthermore,

$$\delta_G(\mathcal{M}^n) \subseteq \sum_{0 \leq i \leq n} \mathcal{M}^i \otimes \mathcal{M}^{n-i} \subseteq \bigcap_{0 \leq i \leq n} (\mathcal{M}^i \otimes K[G] + K[G] \otimes \mathcal{M}^{n-i})$$

and

$$\delta_G(I) \subseteq \bigcap_{n \geq 0} \delta_G(\mathcal{M}^n) \subseteq \bigcap_{n \geq 0} (\mathcal{M}^n \otimes K[G] + K[G] \otimes \mathcal{M}^n) = I \otimes K[G] + K[G] \otimes I.$$

The supersubgroup $G^{[0]}$ is called a *pseudoconnected component* of G . It is clear that G is pseudoconnected iff $G = G^{[0]}$. Additionally, $\text{Dist}(G) = \text{Dist}(G^{[0]})$ and $\text{Lie}(G) = \text{Lie}(G^{[0]})$. The proof of Lemma 9.1 shows that if $N \trianglelefteq G$, then $N^{[0]} \trianglelefteq G$ also. Besides, an epimorphic image of a pseudoconnected supergroup is again pseudoconnected.

Lemma 9.2 (*Krull's intersection theorem*) *Let A be a finitely generated commutative superalgebra and V be a finitely generated A -supermodule. For any superideal I of A we have $\bigcap_{t \geq 0} I^t V = \{v \in V \mid \text{there exists } x \in I_0 \text{ such that } (1 - x)v = 0\}$.*

Proof. Observe that V is finitely generated as a A_0 -module. Since $I^t \subseteq I_0^{[\frac{t}{2}]} \oplus I_0^{[\frac{t-1}{2}]} I_1$ we see that $I^t V \subseteq I_0^{[\frac{t-1}{2}]} V_0 \oplus I_0^{[\frac{t-1}{2}]} V_1$. Proposition 5, [3], III, §3, concludes the proof.

Proposition 9.1 *Let $\pi : G \rightarrow H$ is an epimorphism of algebraic supergroups. If $\text{char} K = 0$, then the induced short sequence of Lie superalgebras*

$$0 \rightarrow \text{Lie}(\ker \pi) \rightarrow \text{Lie}(G) \xrightarrow{d\pi} \text{Lie}(H) \rightarrow 0$$

is exact.

Proof. Since $I_{\ker \pi} = K[G]\pi^*(\ker \epsilon_H)$ it obviously implies that $\ker d\pi = \text{Lie}(\ker \pi)$. Combining Lemma 9.2 with word-by-word repeating the proof of Proposition 7.6, [4], part I, we obtain that $d\pi : \text{Dist}(G) \rightarrow \text{Dist}(H)$ is surjective. Now Lemma 3.1 concludes the proof.

Remark 9.1 *If $\text{char} K = 0$, then any algebraic supergroup G is reduced (or smooth), meaning that the radical of $K[G]$ coincides with $I_{G_{ev}}$. Indeed, by Theorem 11.4 of [27] the Hopf superalgebra $K[G]/I_{G_{ev}}$ is reduced, and in particular, $G_r = G_{ev}$. It is proved in [28] that a completion of $K[G]$ with respect to a \mathcal{M} -adic topology is isomorphic to $K[[t_1, \dots, t_m | z_1, \dots, z_n]]$ for any $\mathcal{M} \in \text{Max}(K[G])$, where $m = \dim \text{Lie}(G)_0, n = \dim \text{Lie}(G)_1$ (see also Lemma 3.1 and use automorphism $T_{\mathcal{M}}$ from Theorem 5.1).*

Any (left) G -supermodule V is a $\text{Dist}(G)$ -supermodule via $\phi v = \sum (-1)^{|\phi||v_1|} v_1 \phi(f_2)$ for $\phi \in \text{Dist}(G)$ and $v \in V, \tau_V(v) = \sum v_1 \otimes f_2$, see [24, 25]. If V, V' are G -supermodules, then we have a canonical embedding $\text{Hom}_G(V, V') \subseteq \text{Hom}_{\text{Dist}(G)}(V, V')$.

The proof of the following lemmas is a copy of the proofs of Proposition 7.5, Lemmas 7.15 and 7.16, [4], part I.

Lemma 9.3 *Let H_1, H_2 be supersubgroups of an algebraic supergroup G and H_1 be pseudoconnected. Then $H_1 \subseteq H_2$ is equivalent to $\text{Dist}(H_1) \subseteq \text{Dist}(H_2)$. Additionally, if $\text{char} K = 0$, then $H_1 \subseteq H_2$ is equivalent to $\text{Lie}(H_1) \subseteq \text{Lie}(H_2)$.*

Lemma 9.4 *If G is pseudoconnected and algebraic, then a supersubspace W of a G -supermodule V is a G -supesubmodule iff W is a $\text{Dist}(G)$ -supersubmodule. If $\text{char} K = 0$, then W is a G -supesubmodule iff W is a $\text{Lie}(G)$ -supersubmodule.*

Lemma 9.5 *If G is pseudoconnected and algebraic, then $\text{Hom}_G(V, V') = \text{Hom}_{\text{Dist}(G)}(V, V')$ for any G -supermodules V and V' . If $\text{char} K = 0$, then $\text{Hom}_G(V, V') = \text{Hom}_{\text{Lie}(G)}(V, V')$.*

Proposition 9.2 *If L is algebraic and $N \trianglelefteq L$, then the equality $\text{Lie}(N) = \text{Lie}(L)$ implies that $L/\tilde{\tilde{N}} = L/\tilde{N}$ is affine and finite.*

Proof. Without loss of generality, one can assume that $K[L]^N = K$ and K is algebraically closed. Denote the supergroup $N \cap G_{red}$ by U . We have

$$Lie(U) = Lie(N) \cap Lie(G_{red}) = Lie(G) \cap Lie(G_{red}) = Lie(G_{red}).$$

It follows that the affine group L_r/\tilde{U} is finite. In other words, an algebra $B = K[L_{red}/\tilde{U}] = K[L_{red}]^U$ is finite dimensional. By Lemma 6.2 of [27] we have $B = \prod_{1 \leq i \leq n} Be'_i$, where e'_1, \dots, e'_n are pairwise orthogonal idempotents such that $\sum_{1 \leq i \leq n} e'_i = 1$. Besides, each algebra Be'_i is isomorphic to K . By Corollary 1, [3], II, §4, there are pairwise orthogonal idempotents $e_1, \dots, e_n \in K[L]_0$ such that their respective images in $K[L_{red}]$ coincide with e'_1, \dots, e'_n and $\sum_{1 \leq i \leq n} e_i = 1$. Consider an idempotent $e = e_i$. As in Lemma 6.3 we have $\delta_L(e) - e \otimes 1 \in r \otimes K[L] + K[L] \otimes (r + I_N)$, where $r = radK[L]$. On the other hand, for any odd exponent k , the equality $(\delta_L(e) - e \otimes 1)^k = \delta_L(e) - e \otimes 1$ holds. For sufficiently large (odd) integer k we infer $\delta_L(e) - e \otimes 1 \in K[L] \otimes I_N$, forcing $e \in K$. It obviously implies that $n = 1, e_1 = 1$ and $B = K$. Repeating the arguments from Propositions 6.5 and 6.6, we see that L/\tilde{N} is affine. By Remark 6.1 $R = K[L]^N$ is finitely generated and $L/\tilde{N} = L\tilde{N}$. Let I be a radical of R . Since $K[L]$ is a faithfully flat R -module we have $r \cap R \subseteq I = R \cap K[L]I \subseteq r$, that is $I = R \cap r$. In other words, the induced morphism $L_{red} \rightarrow (L/\tilde{N})_{red}$ is an epimorphism. In particular, $(L/\tilde{N})_{red} \simeq L_{red}/U$ is finite.

Corollary 9.1 *For any algebraic supergroup L the quotient $L/\tilde{L}^{[0]}$ is a finite supergroup.*

Remark 9.2 *If $charK = 0$ and $Lie(G_{ev}) = Lie(N_{ev})$, then all statements of Proposition 9.2 hold also.*

We define a *connected* component of an algebraic group G as the preimage of $(G/\tilde{G}^{[0]})^{(0)}$ in G . It is not difficult to check that $G^{(0)}$ can be defined as the intersection of kernels of all morphisms $G \rightarrow L$, where L is an even etale (super)group. Notice also that for any finite G both components are the same. In fact, if e_1, \dots, e_n are all primitive idempotents of $K[G]$ and $\epsilon_G(e_1) = 1$, then $\ker \epsilon_G = r + K[G]e$, where $e = \sum_{2 \leq i \leq n} e_i$ and $r = radK[G]$. Thus $I_{G^{[0]}} = K[G]e = I_{G^{(0)}}$.

Remark 9.3 *The supersubgroup $G^{(0)}$ is an open subfunctor of G . In fact, if e_1, \dots, e_n are all primitive idempotents of $K[G]^{G^{[0]}}$, then $G^{(0)} = G_{e_1}$, provided $\epsilon_G(e_1) = 1$.*

Question 9.1 *Does $G^{[0]}$ coincide with $G^{(0)}$ for arbitrary G ?*

Question 9.2 *Is $G^{[0]}$ an open subfunctor for arbitrary G ?*

Lemma 9.6 *If G is pseudoconnected or connected, then $Lie(G) = 0$ implies $G = E$. In particular, if $charK = 0$ and G is algebraic, then $G^{(0)} = G^{[0]}$.*

Proof. In the above notations $Lie(G) = 0$ iff $r \subseteq K[G]e$, that is $\ker \epsilon_G = Be$ is an idempotent and nilpotent ideal simultaneously. As for the second statement, Proposition 9.1 implies $Lie(G^{(0)})/G^{[0]} = 0$.

Lemma 9.7 *If H is a supersubgroup of G , then $Lie(N_G(H)) = (Lie(G)/Lie(H))^{\mathbf{Ad}(H)}$.*

Proof. As in [6], II, §5, Lemma 5.7, it is enough to observe that $x \in \mathbf{Lie}(N_G(H))(K)$ iff for any $A \in \mathcal{SAlg}_K$ and $h \in H(A)$ we have

$$e^{\varepsilon_{|x|}x'}h'e^{-\varepsilon_{|x|}x'}h'^{-1} = e^{\varepsilon_{|x|}(x' - \mathbf{Ad}(h)(x'))} \in H(A[\varepsilon_0, \varepsilon_1]),$$

$$e^{-\varepsilon_{|x|}x'}h'e^{\varepsilon_{|x|}x'}h'^{-1} = e^{\varepsilon_{|x|}(\mathbf{Ad}(h)(x') - x')} \in H(A[\varepsilon_0, \varepsilon_1]),$$

where $x' = G(\iota_K^A)(x)$, $h' = G(p_A)(h)$. In other words, $x \in \mathbf{Lie}(N_G(H))(K)$ iff $x' \pm \mathbf{Ad}(h)(x') \in \mathbf{Lie}(H)(A)$ for any superalgebra A .

Let us return to the situation of Section 6, that is $N \leq G$, where G is algebraic. As before, one can assume that K is an algebraically closed of zero characteristic and, if it is necessary, that $K[G]^N = K$. Define the *lower central (solvable) series* of L by $L^1 = L$, $L^{i+1} = [L^i, L]$ (respectively, $L^{(0)} = L$, $L^{(i+1)} = [L^{(i)}, L^{(i)}]$).

As in Section 7 a pair (G, N) is called *bad*, whenever \tilde{G}/N is not affine. A vector $v \in \mathbf{N}^2$ is called *positive* iff at least one coordinate of v is positive. Partially order the set of bad pairs by $(G, N) < (G', N')$ iff $\text{sdim Lie}(G') - \text{sdim Lie}(G)$ is positive, otherwise $\text{sdim Lie}(G') - \text{sdim Lie}(G) = 0$ and $\text{sdim Lie}(N') - \text{sdim Lie}(N)$ is positive. Choose a minimal bad pair (G, N) .

Proposition 9.3 *If the superalgebra $L = \text{Lie}(G)$ is not simple, then it is either semisimple, whose a unique proper ideal is L^2 , or $L^2 = 0$.*

Proof. Let I be an proper ideal of L . Consider L as G -supermodule via $\mathbf{Ad} : G \rightarrow GL(L)$. By Lemma 9.4 I is a G -supersubmodule and we define the induced morphism $\mathbf{Ad}_I : G \rightarrow GL(L/I)$. Denote $\ker \mathbf{Ad}_I$ by H . By Proposition 9.1 $\text{Lie}(H)$ is a proper supersubalgebra of L iff $L^2 \not\subseteq I$. As above, the minimality of (G, N) and Propositions 6.4 and 7.1 imply that L^2 is a smallest (possibly zero) ideal of L . If L is not semisimple, then considering the morphism $\text{Ad}|_I : G \rightarrow GL(I)$ for a proper abelian ideal I , we obtain $[L, I] = 0$. In particular, $L^3 = 0$. Finally, repeating the above arguments for the morphism \mathbf{Ad} we conclude that $L^2 = 0$.

Using Propositions 6.4 and 7.1 and Theorem 7.1 as well, one can always assume that G is connected. If $L^2 = 0$, then $\text{Dist}(G)$ is a commutative superalgebra. In its turn, $K[G]$ is cocommutative and G is an abelian supergroup. In particular, $G_{ev} \trianglelefteq G$ and the minimality arguments imply that $\text{Lie}(G_{ev}) = 0$. Corollary 7.1 and Theorem 7.1 conclude the proof in this case. If L is simple, then $\text{Lie}(N) = 0$. By Proposition 6.4 and Lemma 9.6 this case is reduced to $N = N_{ev}$. The algebra $K[G]_0$ can be regarded as a coordinate algebra of an affine scheme on which N acts on the right. Combining [6], III, §2, n°4, with $K[G]^N = K$ we obtain that $K[G]_0$ is finitely dimensional. By Theorem 7.1 this case is also done.

Finally, if L is semisimple, then we consider the supersubgroup $H = (G_{ev}N)^{[0]}$. Denote by Q the normalizer $N_G(H)$ and by D its Lie superalgebra. By the above, one can assume that $\text{Lie}(N) \neq 0$. It follows that $L^2 \subseteq \text{Lie}(N)$. Denote $\text{Lie}(G_{ev}N) = L_0 + \text{Lie}(N)$ by M . Since $L^2 \subseteq M$ the supergroup H acts trivially on L/M . By Lemma 9.7 $D/M = (L/M)^{\mathbf{Ad}(H)} = L/M$ that infers $G = Q$ and $H \trianglelefteq G$. Again, by the minimality of (G, N) we have either $G = G_{ev}N$ and then $\tilde{G}/N \simeq G_{ev}/(\tilde{G}_{ev} \cap N)$ is affine, or $L_0 \subseteq \text{Lie}(N)$ and Remark 9.2 concludes the proof of Theorem 6.2.

10 Two examples

One more example of not necessary normal but faithfully exact supersubgroup is given by a *Levi supersubgroup*. In notations of Proposition 6.3, a Levi supersubgroup $L_{s,t}$ of $G = GL(m|n)$ consists of all matrices satisfying the equations $A_{12} = 0, A_{21} = 0, B_{12} = 0, B_{21} = 0, B_{22} = 0, C_{12} = 0, C_{21} = 0, C_{22} = 0, D_{12} = 0, D_{21} = 0$. Additionally, the blocks A_{22} and D_{22} are diagonal matrices. It is clear that $L_{s,t} \simeq GL(s|t) \times T$, where T is a torus of dimension $m + n - s - t$. Represent the coordinate superalgebra of $K[GL(m|n)]$ as $K[A, B, C, D]_{d_1 d_2}$, where the blocks A, B, C, D are identified with the sets of their coefficients, and $d_1 = \det(A), d_2 = \det(D)$.

Define the map $\pi : \underline{m+n} \rightarrow \underline{m+n}$ by the rule $\pi(i) = i + n$ and $\pi(j) = j - m$ for $1 \leq i \leq m < j \leq m + n$.

Lemma 10.1 *There is a canonical isomorphism $\psi : GL(m|n) \rightarrow GL(n|m)$ such that $\psi(L_{s,t}) = L_{t,s}$.*

Proof. Denote the matrix coordinate functions on $GL(m|n)$ by a_{ij} and the similar functions on $GL(n|m)$ by a'_{ij} . It is easy to see that $\psi^*(a_{ij}) = a'_{\pi(i), \pi(j)}$ induces the required Hopf superalgebra isomorphism $K[GL(m|n)] \simeq K[GL(n|m)]$. In fact, only the equality $\psi^* s_{GL(m|n)} = s_{GL(n|m)} \psi^*$ is not trivial. It is enough to prove it for generators a_{ij} and using the following formulae

$$\psi^* \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \psi^*(D) & \psi^*(C) \\ \psi^*(B) & \psi^*(A) \end{pmatrix},$$

$$s_{GL(m|n)} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

it can be done by straightforward calculations.

Using Lemma 10.1 and transitivity of inducing functor it remains to prove that $L_{m, n-1}$ (or $L_{m-1, n}$) is faithfully exact. In what follows let $G = GL(m|n)$.

Lemma 10.2 *The elements $a_{i, m+n} s_G(a_{m+n, j})$ for $1 \leq i, j \leq m + n$, are $L = L_{m, n-1}$ -invariants. Moreover, they generate a left coideal supersubalgebra R such that $I_L = K[G]R^+$.*

Proof. Denote the image of $x \in K[G]$ in $K[L]$ by \overline{x} . Since

$$\delta_G(a_{i, m+n} s_G(a_{m+n, j})) = \sum_{1 \leq t, l \leq m+n} (-1)^{|a_{lj}| |a_{m+n, l}| + |a_{t, m+n}| |a_{lj}|} a_{it} s_G(a_{lj}) \otimes a_{t, m+n} s_G(a_{m+n, l}),$$

it follows that R is a left coideal. Considering $K[G]$ as a $K[L]$ -supercomodule (where L acts on G by the right multiplication) we determine

$$\tau_{K[G]}(a_{i, m+n} s_G(a_{m+n, j})) = a_{i, m+n} s_G(a_{m+n, j}) \otimes \overline{a_{m+n, m+n} s_G(a_{m+n, m+n})} =$$

$$a_{i, m+n} s_G(a_{m+n, j}) \otimes 1.$$

Notice that $a_{i, m+n} s_G(a_{m+n, j}) - \delta_{i, m+n} \delta_{m+n, j} \in I_L$ for $1 \leq i, j \leq m + n$ and consider an index $i \neq m + n$. We have $a_{i, m+n} = \sum_{1 \leq j \leq m+n} a_{i, m+n} s_G(a_{m+n, j}) a_{j, m+n} \in I_L$ and

symmetrically, $s_G(a_{m+n, j}) \in I_L$ for $j \neq m+n$. Denote by I a superideal generated by elements $a_{i, m+n}, s_G(a_{m+n, j})$ for $1 \leq i, j < m+n$. It is obvious that $L \subseteq V(I)$. On the other hand, $g \in V(I)(A)$ iff $g \in \text{Stab}_G(Kv_{m+n})(A)$ and $g^{-1} \in \text{Stab}_G(\sum_{1 \leq i \leq m+n-1} Kv_i)(A)$. The superversion of [4], part I (1.4) completes the proof of this claim.

By Proposition 5.2 and Theorem 5.1 all we have to show is that $K[G]\mathcal{M} \neq K[G]$ for all $\mathcal{M} \in \text{Max}(R)$. Using the reduction from Corollary 5.1 one can work with algebraic groups $L_{0, n-1} \subseteq GL(0|n) \simeq GL(n)$. In other words, we can set $G = GL(n)$, $L = \text{Stab}_G(Kv_n) \cap \text{Stab}_G(\sum_{1 \leq i \leq n-1} Kv_i)$. By Corollary 4.5 of [21], the quotient \tilde{G}/L is affine and it is isomorphic to $Sp_K K[G]^L$. In particular, $K[G]^L \leq K[G]$. We will show that $K[G]^L$ is generated by the elements $a_{in}s_G(a_{nj})$ for $1 \leq i, j \leq n$.

Consider an element $\frac{f}{d^k} \in K[G]$, where $f \in K[a_{ij} | 1 \leq i, j \leq n]$ and $d = \det(a_{ij})$. Represent f as

$$f = \sum_{\alpha \in \mathbf{N}^n} f_\alpha \prod_{1 \leq i \leq n} a_{in}^{\alpha_i}, \text{ where } f_\alpha \in K[a_{ij} | 1 \leq i \leq n, 1 \leq j \leq n-1].$$

It can be checked easily that $\frac{f}{d^k} \in K[G]^L$ iff all monomials $\prod_{1 \leq i \leq n} a_{in}^{\alpha_i}$ in the above representation of f have degree k and all coefficients f_α are $GL(n-1)$ -semi-invariants of weight k with respect to the action of this group by right multiplications on the variety of $n \times (n-1)$ -matrices $M_{n \times (n-1)}$. By Igusa's Theorem (see [30] and Corollary 3.5 of [31]) the algebra $K[M_{n \times (n-1)}]^{SL(n-1)} = K[a_{ij} | 1 \leq i \leq n, 1 \leq j \leq n-1]^{SL(n-1)}$ is generated by all minors of size $n-1$ which are just semi-invariants $s_G(a_{nj})d$ of weight 1.

The next example shows that a quotient \tilde{X}/G (as well X/\tilde{G}) is not always affine even if G is finite. In what follows $\text{char} K = 0$. Let $G = G_a^-$ be an odd unipotent supergroup such that $K[G] = K[t]$, where $|t| = 1$, $\delta_G(t) = t \otimes 1 + 1 \otimes t$ and $\epsilon_G(t) = 0$, $s_G(t) = -t$. A superspace V is a G -supermodule iff there is $\phi \in \text{End}_K(V)_1$, $\phi^2 = 0$, such that $\tau_V(v) = v \otimes 1 + \phi(v) \otimes t$. Moreover, $V^G = \ker \phi$. Assume that $\text{sdim } V = (1, 1)$ and v_1, v_2 form a \mathbf{Z}_2 -homogeneous basis of V . Set $\phi(v_1) = v_2, \phi(v_2) = 0$. The symmetric superalgebra $S(V)$ has the induced G -supermodule structure by

$$\tau_{S(V)}(v_1^r) = v_1^r \otimes 1 + r v_1^{r-1} v_2 \otimes t, \quad \tau_{S(V)}(v_1^{r-1} v_2) = v_1^{r-1} v_2 \otimes 1, r \geq 0.$$

Since $\tau_{S(V)}$ is a superalgebra morphism, there is an affine scheme X such that $K[X] = S(V)$ and G acts on X . Denote $K[X]^G$ by R . We have $R_0 = K, R_1 = \sum_{t \geq 0} K v_1^t v_2$ and $R_1^2 = 0$. In particular, R is commutative as an algebra.

Lemma 10.3 *Any flat R -supermodule is flat as a module.*

Proof. Any exact sequence of R -modules $0 \rightarrow V \rightarrow W \rightarrow U \rightarrow 0$ can be turned into an exact sequence of R -supermodules. In fact, set $V_1 = V \cap R_1 W$ and let V_0 be a complement of V_1 to V . Since $V_0 \cap R_1 W = 0$, then $R_1 W$ has a complement W_0 to W such that $V_0 \subseteq W_0$. Finally, $U = W_0/V_0 \oplus W_1/V_1$. If M is a flat R -supermodule, then the functor $M \otimes_R ?$ takes our sequence (of supermodules!) to an exact one.

Assume that \tilde{X}/G or X/\tilde{G} is an affine superscheme. By Proposition 4.1 $S(V)$ is a faithfully flat R -supermodule. Combining Lemma 10.3 with Corollary 2.1 [3], II, §3, we obtain that $S(V)$ is a free R -module. Let $Rf \simeq R, f \in S(V)$. The equality $Rf = Rf_0 + Kf_1$ implies that either $Rf = Rf_0$ or Kf_1 is a projective R -module. Again by Corollary 2.1

[3], II, §3, Kf_1 has to be free, but it is obviously impossible. Thus $S(V) = \bigoplus Rf$, where f runs over a basis of $S(V)_0$. Without loss of generality one can assume that some f equals 1. On the other hand, the summand R has nontrivial intersections with all other summands! This contradiction shows that both X/\tilde{G} and $X/\tilde{\tilde{G}}$ are not affine.

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